Problems: Channel Estimation and Equalization
ECE-GY 6023. Wireless Communications

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1. *Kernel regression:* Consider a kernel regression estimate of the form

$$\widehat{h}[n] = \frac{\sum_{\ell \in I} w_\ell \widehat{h}_0[n+\ell]}{\sum_{\ell \in I} w_\ell}, \tag{1}$$

where I is a set of reference symbol locations, $\widehat{h}_0[n]$ are raw estimates of the channel on the reference symbols and w_ℓ is a kernel. Suppose you use a triangular kernel,

$$w_\ell = \max\{1 - \ell/L, 0\},$$

for some width L and the reference symbols are spaced every d positions,

$$I = \{0, d, 2d, \quad , Md\}.$$

Assume $L = 12$, $d = 4$ and $M = 10$. For each value n, below write $\widehat{h}[n]$ as a linear combination of the values $\widehat{h}_0[k]$.

   (a) The symbol on the edge, $n = 0$.

   (b) A symbol in the middle located on a reference symbol location, $n = 20$.

   (c) A symbol in the middle located between two reference symbol locations, $n = 22$.

2. *Channel estimation error for a stochastic model:* Suppose that we form an estimate,

$$\widehat{h}[n] = \frac{1}{2}\left[\widehat{h}_0[n-L] + \widehat{h}_0[n+L]\right].$$

with measurements

$$\widehat{h}_0[k] = h[k] + v[k], \quad v[k] \sim \mathcal{CN}(0, N_v).$$

Suppose we can model $h[k]$ as a wide sense stationary Gaussian random process with auto-correlation $R[k] = \mathbb{E}(h[n]h[n+k]^*)$.

   (a) Find the mean squared error,

$$\epsilon = \mathbb{E}|\widehat{h}[n] - h[n]|^2$$

   in terms of the auto-correlation $R[n]$ and the error in the raw channel estimate N_v.

(b) If $h[n]$ is a narrowband fading process with Jake's spectrum then its auto-correlation is

$$R[k] = E_s J_0(2\pi f_{\max} kT)$$

where E_s is the energy per sample, $f_{\max}$ is the maximum Doppler spread and T is the sample period. Write and plot the normalized MSE, ϵ/E_s as a function of $f_{\max}LT$ for the case when $N_v = 0$. Note that when there is zero noise, MSE, represents the bias squared.

(c) In the above model, there is one reference symbol every $2L$ samples, so the overhead is $1/(2L)$. Suppose that $f_{\max} = 100\,\text{Hz}$, and the sample period is $T = 1\,\mu\text{s}$. Using the model in part (b), what is the minimum overhead if we need the MSE, ϵ, to be less than $20\,\text{dB}$ below E_s?

3. *Joint Likelihoods*: Suppose that we have two BPSK symbols, $i = 0, 1$:

$$r_i = hx_i + w_i, \quad w_i \sim \mathcal{CN}(0, N_0), \quad x_i = \pm 1,$$

where the noise w_i is i.i.d. The channel gain h is unknown and can be modeled as a complex Gaussian $h \sim \mathcal{CN}(0, E_s)$. We use the first symbol, $x_0 = 1$, as a reference symbol. So, there are two possibilities:

$$\mathbf{x} = \mathbf{x}^{(1)} = (1, 1) \text{ or } \mathbf{x}^{(2)} = (1, -1).$$

We will compute the LLR for these two possibilties.

(a) What is the mean and covariance matrix of the vector $\mathbf{r} = (r_0, r_1)$ for the two values of $\mathbf{x}$?

(b) Using the fact that $\mathbf{r}$ is a Gaussian random vector, what is the log likelihood ratio

$$\text{LLR} = \log \left[\frac{p(\mathbf{r}|\mathbf{x} = (1, 1))}{p(\mathbf{r}|\mathbf{x} = (1, -1))} \right].$$

You can simplify the expression with the matrix identity,

$$(\mathbf{I} + \gamma \mathbf{u}\mathbf{u}^*)^{-1} = \mathbf{I} - \frac{\gamma}{1 + \gamma \|\mathbf{u}\|^2} \mathbf{u}\mathbf{u}^*$$

for any vector $\mathbf{u}$.