Unit 7. Channel Estimation and Equalization

ECE-GY 6023. WIRELESS COMMUNICATIONS

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Learning Objectives

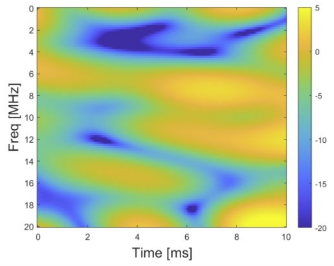
- Describe role of channel estimation in the receiver steps
- Describe role and configuration of reference signals in OFDM
  - Compute overhead
  - Qualitatively relate density to the coherence time and bandwidth
- Implement and simulate simple kernel-type channel estimators
- Compute bias and variance of the kernel estimators
- Optimize the kernel to minimize the bias + variance error
- Compute the signal to distortion and noise ratio (SNDR) for a given estimator
- Measure the SNDR via simulation
- Implement a simple channel estimator and equalizer for 5G NR downlink

Outline

- Role of Channel Estimation and Equalization
- OFDM Channel Estimation via Kernel Regression
- Bias-Variance Tradeoff

Channel Estimation and Equalization

$x[n,k]$ →



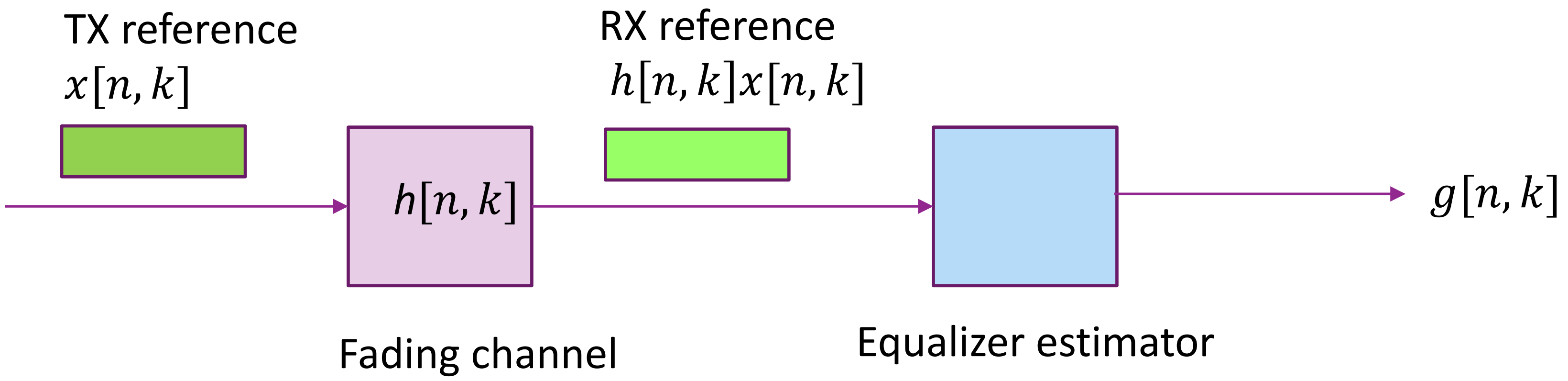
Fading channel $h[n,k]$

→ $r[n,k]$ → Equalization → $\hat{x}[n,k] \approx x[n,k]$

- Channels fade in time and frequency
  - In OFDM, can be described by time-varying frequency response: $r[n,k] = h[n,k]x[n,k] + w[n,k]$
- Equalizers: Estimate TX symbols from faded symbols
  - Approximately invert the channel $\hat{x}[k,n] \approx \frac{r[n,k]}{h[n,k]}$ or MMSE $\hat{x}[k,n] = \frac{h[n,k]^* r[n,k]}{|h[n,k]|^2 + N_0}$

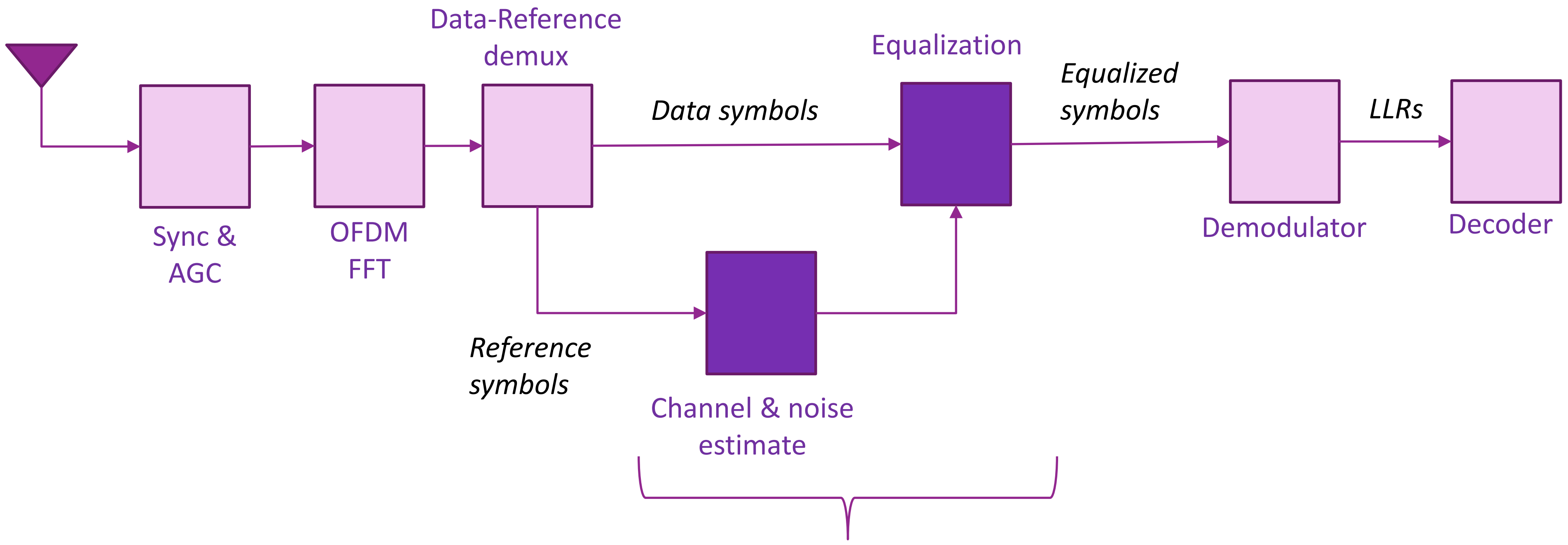
Key challenge: Equalizers need a channel and/or noise estimate

Reference Symbols



- ❑Equalizers require training or reference signals
- ❑TX sends a known reference sequence $x[n, k]$ on some locations (k, n)
- ❑RX sees $r[n, k] = h[n, k]x[n, k] + w[n, k]$
- ❑Estimates channel $h[n, k]$ and/or equalizer $g[n, k]$ for $r[n, k]$

Receiver Steps in OFDM



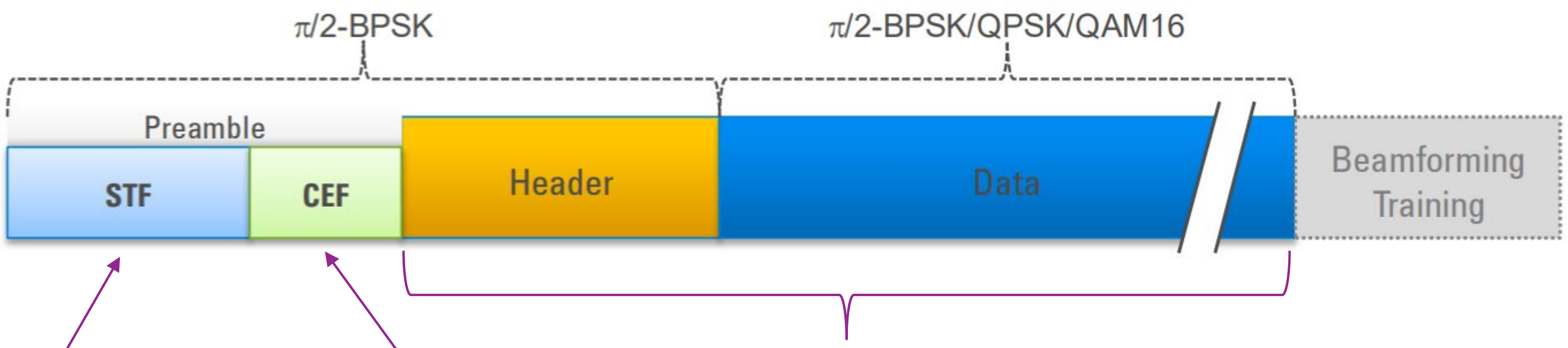
Today’s unit

Frequency-Domain and Time-Domain EQ

- Frequency-Domain Equalization:
  - Typically, an OFDM channel $r[n,k] = h[n,k]x[n,k] + w[n,k]$
  - Estimate frequency-domain channel $h[n,k]$
  - Perform equalization with a filter $\hat{x}[n] = g[n,k]r[n,k]$
  - Used in 4G, 5G OFDM systems, most 802.11 versions

- Time-Domain Equalization:
  - Model channel in time-domain $r[n] = \sum_k h[k]\, x[n-k] + w[n]$
  - Estimate channel impulse response
  - Perform equalization with a filter $\hat{x}[n] = \sum_k g[k]\, r[n-k]$
  - Used in 2G, 3G CDMA systems, 802.11ad single carrier version
  - Do not discuss in this unit (except next few slides)

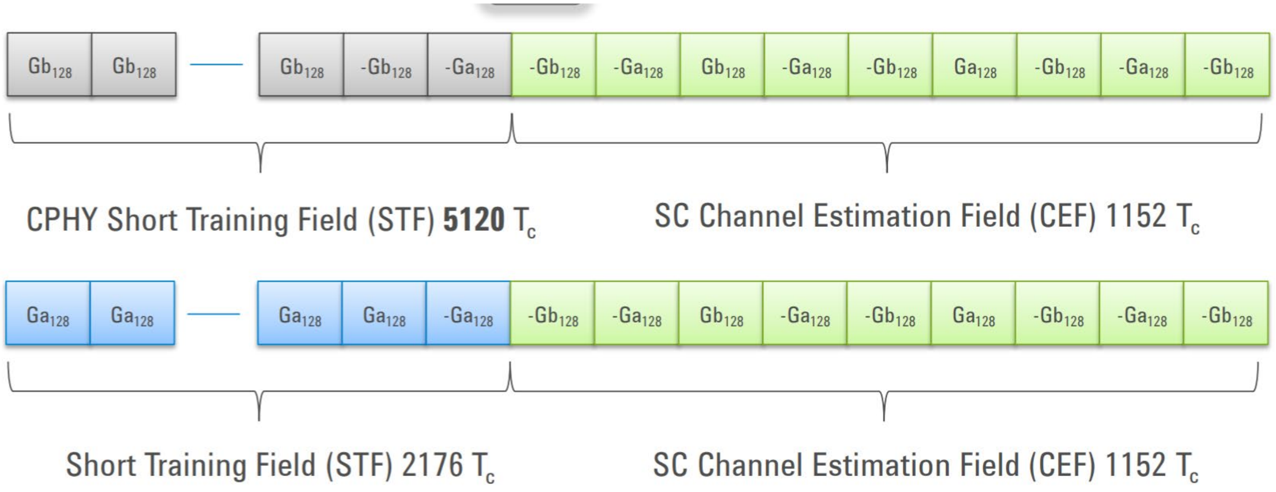
802.11 ad Time-Domain Equalizer



802.11ad uses time-domain equalization

- STF: Short training field
  - AGC, detection of packet
  - coarse estimate of timing
  - Process as previous lecture
- CEF: Channel estimation field
  - Train equalizer

802.11ad Preamble Details



Control packets

Longer STF.

Need to be decoded by all stations. No directional gain

Data packets

Shorter STF

- Based on complementary Golay codes $[Ga, Gb]$
- Have low auto-correlation
- $\frac{1}{T_c} = 1.76$ Gsamp/S
- For data packets: STF = 1.23 μs, CE= 0.654 μs

Outline

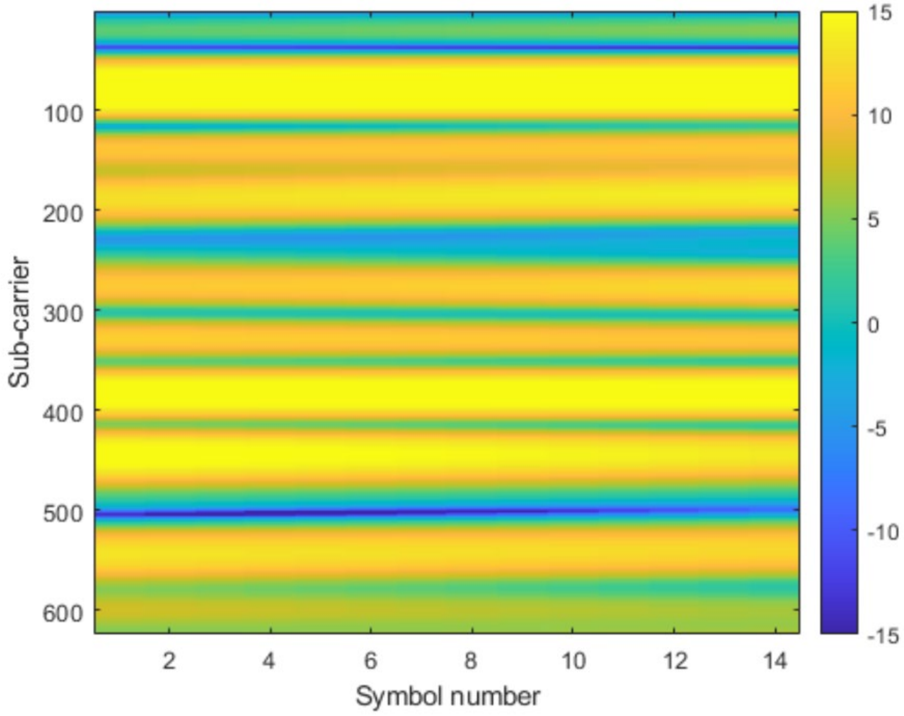
- ❑ Role of Channel Estimation and Equalization
- ❑ OFDM Channel Estimation via Kernel Regression
- ❑ Bias-Variance Tradeoff

OFDM Channel Estimation

- OFDM Channel

$$R[n,k] = H[n,k]X[n,k] + W[n,k]$$

  - Symbol k and sub-carrier n
- Channel $H[n,k]$ varies
  - Over time k due to Doppler spread
  - Over frequency n due to delay spread
- Problem: Estimate $H[n,k]$
- Estimate needed to demodulate data $X[n,k]$



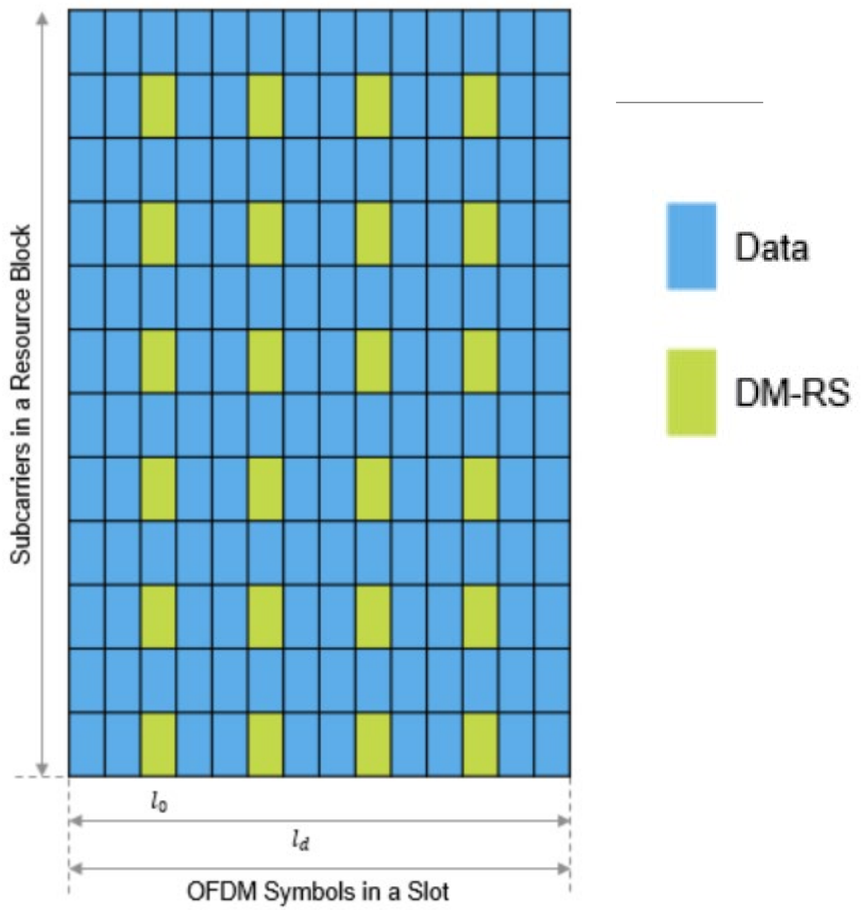
51 RB, 15 kHz SCS, $\sim$10 MHz bandwidth NR slot
500 ns delay spread, 200 Hz Doppler, $\pm 60°$ AoA

Reference Signals

- OFDM Channel

$$R[n,k] = H[n,k]X[n,k] + W[n,k]$$

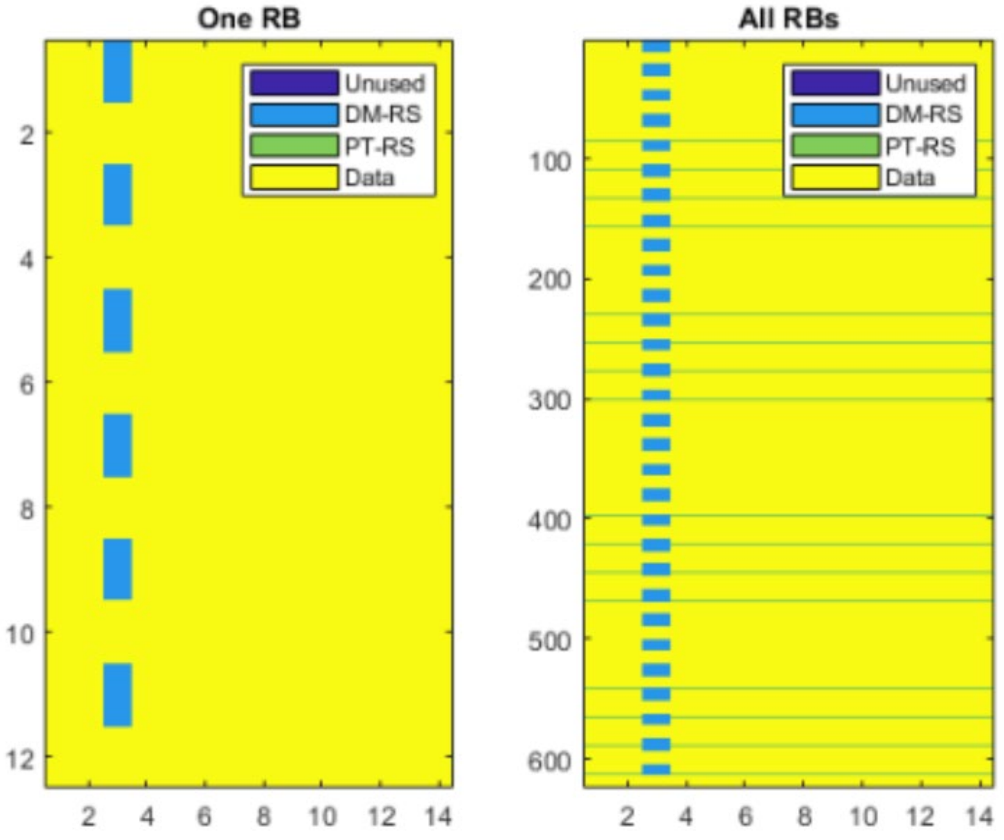
- 5G NR Demodulation Reference Signals
  - Called DM-RS
  - Transmitted in downlink with data (PDCSH)
  - Very configurable in terms of over-head
- Basic channel estimation idea
  - Transmit know $X[n,k]$ on DM-RS.
  - Estimate $\widehat{H}_0[n,k] \approx R[n,k]/X[n,k]$ on DM-RS symbols
  - Interpolate to data symbols



a) 3 Additional Positions for Single-Symbol

Configuring in DM-RS

- ❑Example configuration
  - DM-RS on one symbol in RB
  - Occupies 6 out of 12 REs
- ❑Also added a phase-tracking reference signal
  - Tracks phase variations over time
  - Useful for carrier frequency offset and phase noise (more on this later)
- ❑Many configurations possible
- ❑See excellent MATLAB 5G tutorial with videos



Configuring in MATLAB

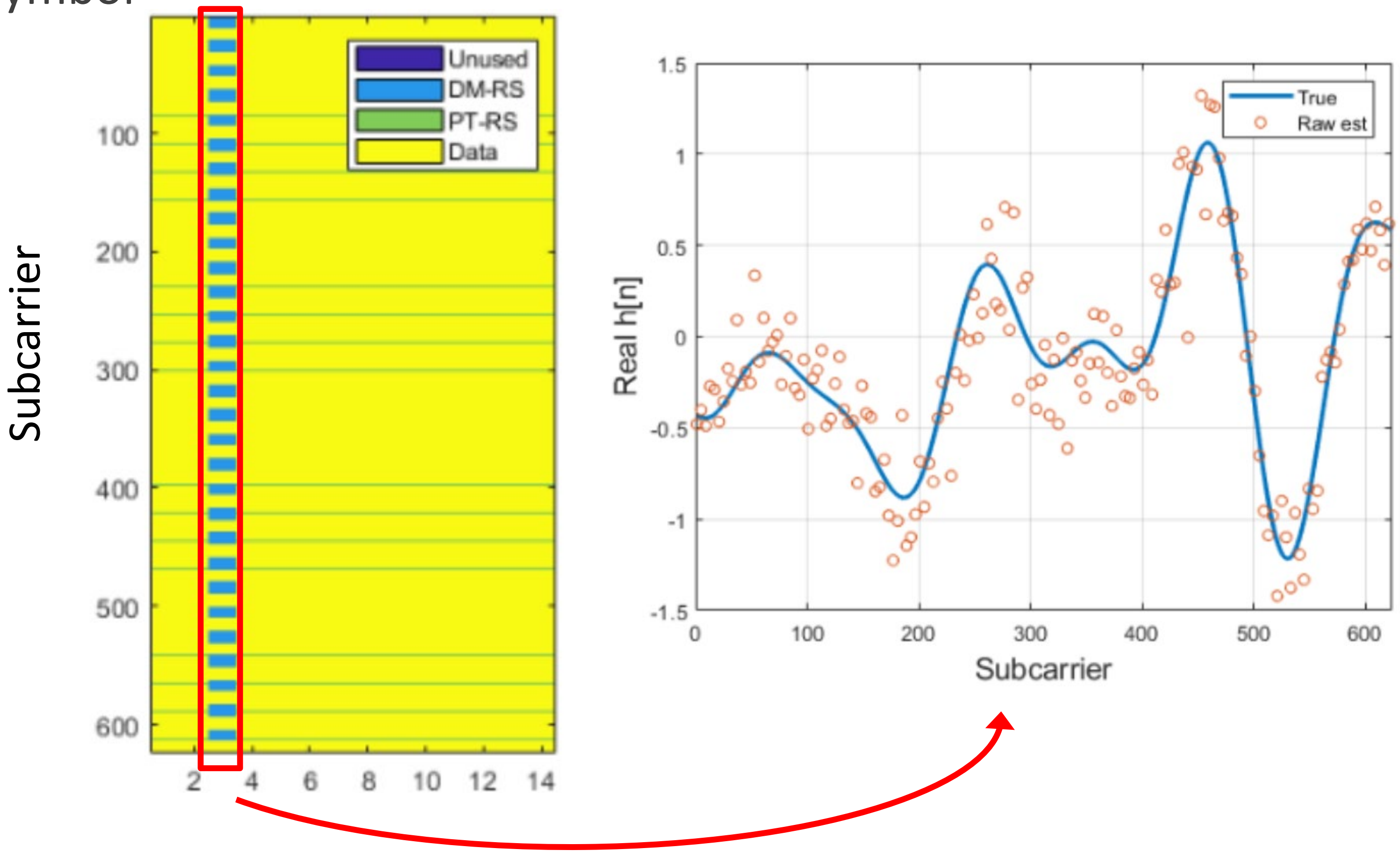
- MATLAB 5G Toolbox
  - Provides all config possibilities
  - Easy to use
- See [excellent MATLAB 5G tutorial]() with videos:

```
% DM-RS config
dmrsConfig = nrPDSCHDMRSConfig(...
     'NumCDMGroupsWithoutData', 1, ...   % No unused DM-RS
     'DMRSAdditionalPosition', 0, ...    % No additional DM-RS in time
     'DMRSConfigurationType', 1);  % 1=6 DM-RS per sym,  2=4 per sym
pdschConfig.DMRS = dmrsConfig;

% Get the PT-RS symbols and indices
ptrsSym = nrPDSCHPTRS(carrierConfig,pdschConfig);
ptrsInd = nrPDSCHPTRSIndices(carrierConfig,pdschConfig);
```

```
% PDSCH DM-RS precoding and mapping
dmrsSym = nrPDSCHDMRS(carrierConfig,pdschConfig);
dmrsInd = nrPDSCHDMRSIndices(carrierConfig,pdschConfig);
```

Estimating Channel in a Single Symbol

- First consider estimation in one symbol
- True channel is $h[n]$
  - $n =$ sub-carrier index
- RX: $r[n] = h[n]x[n] + w[n]$
- DM-RS $x[n]$ known
- Example to right:
  - 10 MHz bandwidth
  - 200 ns delay spread
  - 10 dB SNR on DM-RS

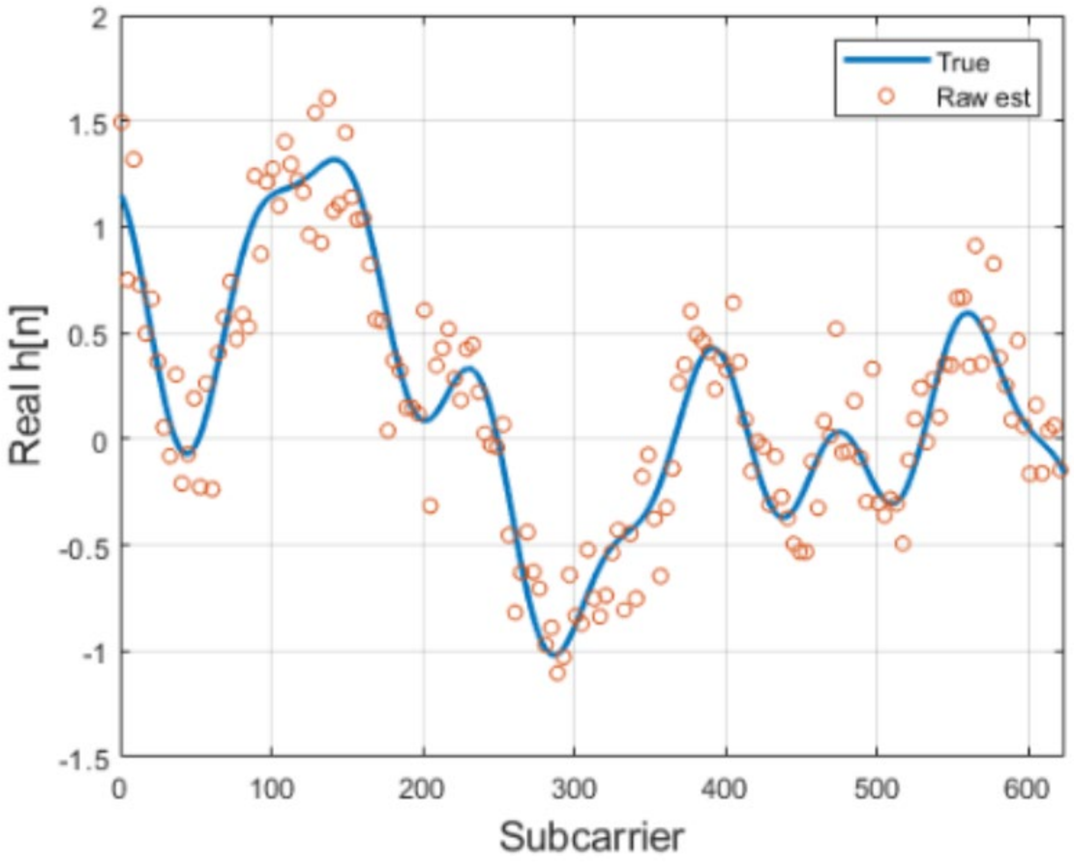


Raw Channel Estimate

- Channel: $r[n] = h[n]x[n] + w[n]$
- On DM-RS positions, $x[n]$ is known
- Raw channel estimate:

$$\hat{h}_0[n] = \frac{r[n]}{x[n]}$$

  - Can be computed in every DM-RS position
- We see estimate is noisy



Raw Channel Estimate Error

- Drop sub-carrier index (since we are looking at one RE)
- Channel: $r = hx + w,\ w \sim CN(0, N_0), |x|^2 = E_x$
- Raw channel estimate: $\hat{h}_0 = \frac{r}{x} = h + \frac{w}{x}$
- Error is

$$E\left|h - \hat{h}_0\right|^2 = \frac{1}{|x|^2} E|w|^2 = \frac{N_0}{E_x}$$

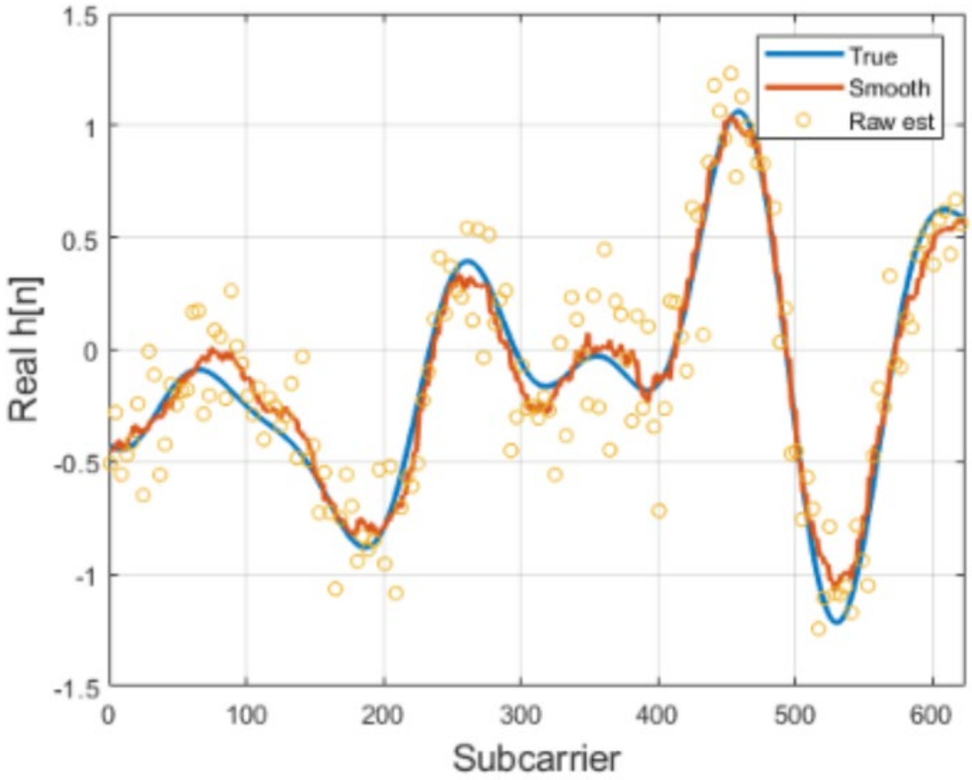
- Relative error is:

$$\frac{E\left|h - \hat{h}_0\right|^2}{|h|^2} = \frac{N_0}{E_x|h|^2} = \frac{1}{\gamma}$$

  - γ =SNR on the symbol
- Relative error is the inverse of the SNR

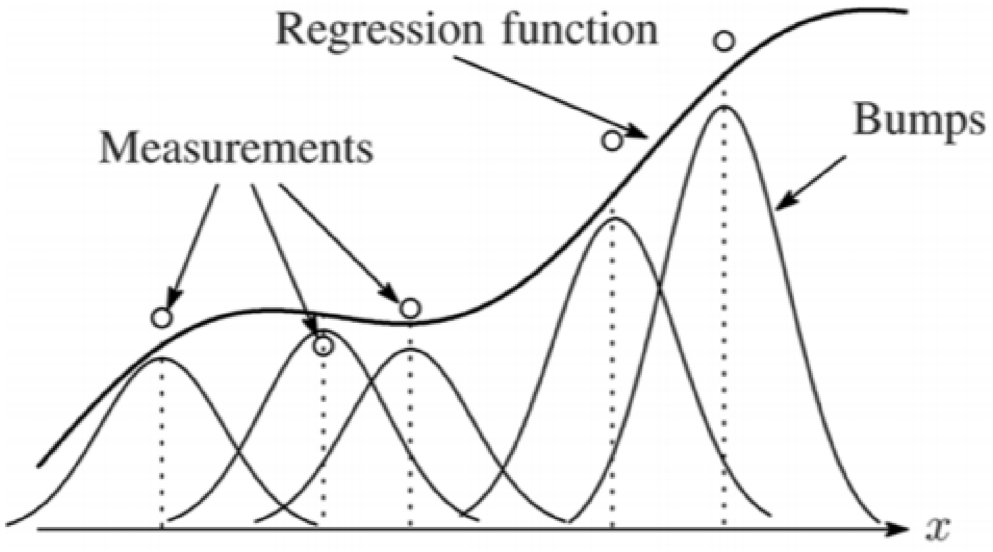
Smoothing the Estimate

- Raw channel estimates are:
  - Noisy
  - Only available on DM-RS locations
- Often smooth the estimate
  - Fit over all locations
- Many options
  - Linear interpolation, splines
  - Parametric
- We look at Kernel regression
  - Works well
  - Easy to implement and analyze



Kernel Regression

- **Problem**: Want to estimate $h[n], n = 1, \ldots, N$
  - Given noisy values $\hat{h}_0[n] = h[n] + w[n]$
  - At locations $n \in I \subset \{0, \ldots., N-1\}$
- **Kernel function** $w[\ell]$
  - Example radial basis function: $w[\ell] = e^{-\ell^2/(2\sigma^2)}$
  - $\sigma =$ "Bandwidth"
- **Kernel estimate**: $\hat{h}[n] = \frac{z_1[n]}{z_0[n]}$
  - $z_1[n] = \sum_{k \in I} w[n-k]\hat{h}_0[k]$,
  - $z_0[n] = \sum_{k \in I} w[n-k]$
- Weights raw samples by the distance $w[n-k]$

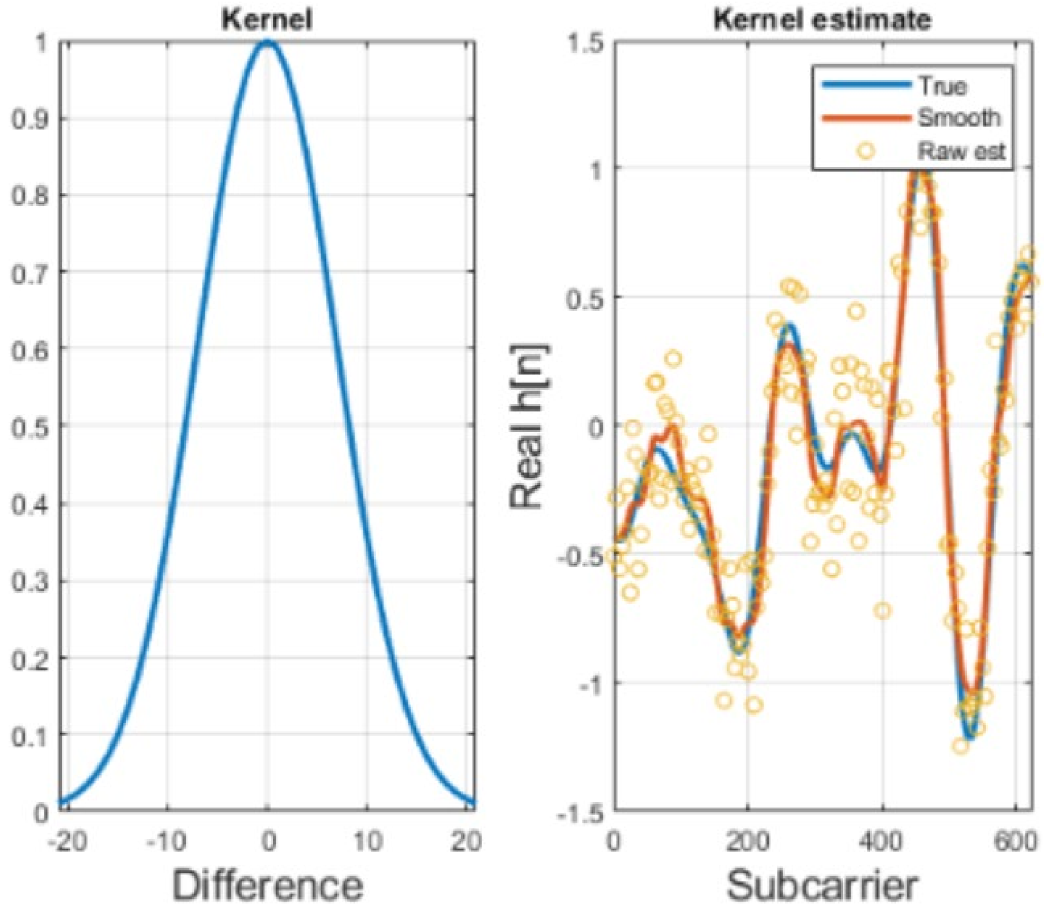


(a) Kernel regression

Takeda, Farisu, Milanfar, Kernel Regression for Image Processing and Reconstruction, 2007

Example

- RBF Kernel
  - Bandwidth $\sigma = 7$
  - Length $L = 3\sigma$ on each side
- Estimate is visibly smoother



MATLAB Implementation

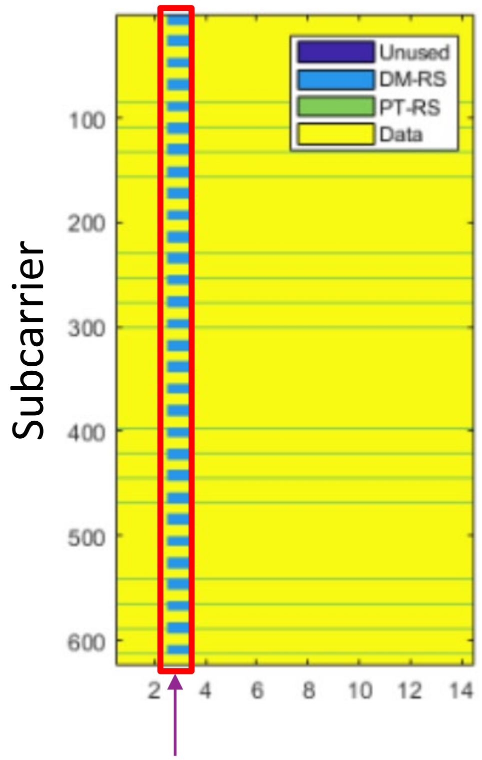
- Easy to implement in software or hardware
- See demo

```
% Create random channel
h = randChan(nsc, dlymean, SubcarrierSpacing);

% Add noise
wvar = db2pow(-snr);
rx = h.*tx + sqrt(wvar/2)*(randn(nsc,1) + 1i*randn(nsc,1));

% Get the raw channel estimates on the RS
hestRaw = rx(rsInd)./rsSym;

% Get channel estimate
hest = kernelReg(rsInd, hestRaw, nsc, len, sig);
```

```
function [hest, w] = kernelReg(ind,hestRaw,nsc,len,sig)
% kernelReg:  Kernel regression with a RBF

% Create the RBF kernel
w = exp(-0.5*(-len:len).^2/sig^2)';

% Place the raw channel esimates in a vector at the locations
% of the indices.
%   y(ind(i))  = hestRaw(i)
%   y0(ind(i)) = 1
y = zeros(nsc,1);
y0 = zeros(nsc,1);
y(ind) = hestRaw;
y0(ind) = 1;

% Get the filter length
len = floor(length(w)/2);

% Filter both raw estimates and the indicators
[z1, z1f] = filter(w,1,y);
z1 = [z1(len+1:end); z1f(1:len)];
[z0, z0f] = filter(w,1,y0);
z0 = [z0(len+1:end); z0f(1:len)];

% Compute the channel estimate
hest = z1./max(1e-8,z0);
end
```

Extending the Estimate to Other Symbols

- Up to now:
  - Estimated at channel at a single OFDM symbol
  - $\widehat{H}[n, k_0]$ on OFDM symbol $k = k_0$
- Can be extended to other symbols in slot:
  - $\widehat{H}[n, k] = \widehat{H}[n, k_0]$
  - Assumes channel is constant over time
  - Coherence time $\gg$ Slot time
- Can also have multiple RS symbols
  - Useful when there is time variations in slot
- See lab



Channel estimate at $k = k_0$

In-Class Exercise

In-Class Exercise

For each channel, find the fraction of the REs used by the channel.

Roughly, estimate the number of DM-RS symbols per coherence BW

Outline

- Role of Channel Estimation and Equalization
- OFDM Channel Estimation via Kernel Regression
- Bias-Variance Tradeoff

Mean Squared Error with Kernel Regression

- Problem: True channel $h[n], n = 1, \ldots, N$
  - Given noisy values $\hat{h}_0[n] = h[n] + v[n]$ at sub-carriers $n \in I$.
- Kernel estimate: $\hat{h}[n] = \frac{z_1[n]}{z_0[n]}$
  - $z_1[n] = \sum_{k \in I} w[n-k] \hat{h}_0[k]$, $z_0[n] = \sum_{k \in I} w[n-k]$
- Mean-squared error:

$$MSE[n] = E\left|\hat{h}[n] - h[n]\right|^2$$

  - Measures the accuracy of the estimate
  - For now, model $h[n]$ as deterministic
  - Average over noise $v[n]$
- May also be interested in the normalized MSE: $NMSE[n] = \frac{E\left|\hat{h}[n] - h[n]\right|^2}{|h[n]|^2}$

Bias and Variance Error

- Error: $e[n] = h[n] - \hat{h}[n]$
- Since $\hat{h}[n]$ is linear in $\hat{h}_0$ can write $e[n] = e_s[n] - e_v[n]$
  - Signal component: $e_s[n] = h[n] - \frac{1}{z_0[n]} \sum_{k \in I} w[n-k] h[k]$
  - Noise component: $e_v[n] = \frac{1}{z_0[n]} \sum_{k \in I} w[n-k] v[k]$
- MSE can be written as two components:

$$\boxed{MSE[n] = Bias^2[\mathrm{n}] + Var[n]}$$

  - Bias error: $Bias^2[\mathrm{n}] = |e_s[n]|^2$
  - Variance error: $Var[n] = E|e_v[n]|^2$

Variance Error

- Noise term $e_v[n] = \frac{1}{z_0[n]} \sum_{k \in I} w[n-k] v[k]$
- Recall $E|v[n]|^2 = \frac{N_0}{E_x}$,
  - N_0 = noise on the reference signals, E_x = Reference signal energy
- Variance on the kernel estimate is:

$$\boxed{Var[n] = E|e_v[n]|^2 = \frac{\sum_{k \in I} |w[n-k]|^2}{\left|\sum_{k \in I} w[n-k]\right|^2} \frac{N_0}{E_x}}$$

- Proof:
  - Since $v[n]$ are uncorrelated, $E|e_v[n]|^2 = \frac{\sum_{k \in I} |w[n-k]|^2 E|v[k]|^2}{|z_0[n]|^2}$
  - Substitute expressions for $z_0[n] = \sum_{k \in I} w[n-k]$ and $E|v[k]|^2 = \frac{N_0}{E_x}$

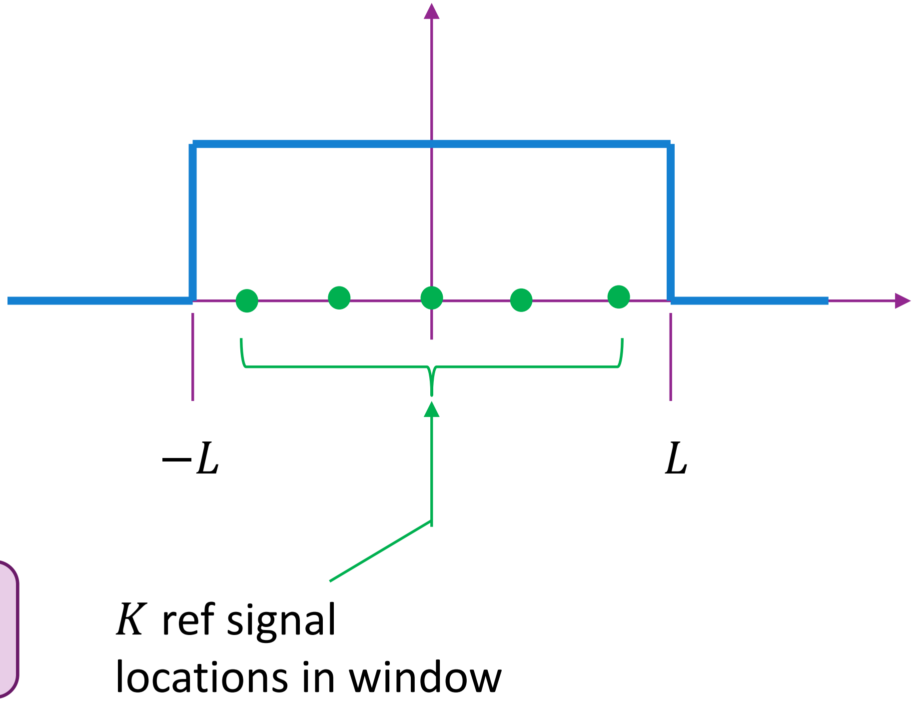
Variance Error for a Uniform Kernel

- Example: Uniform kernel

$$w[\ell] = \begin{cases} 1 & \text{if } |\ell| \le L \\ 0 & \text{else} \end{cases}$$

  - Length $2L+1$
- Suppose that there are K reference signals in window
- Variance error: $Var[n] = \frac{N_0}{KE_x}$
  - Why? $Var[n] = E|e_v[n]|^2 = \frac{\sum_{k\in I} |w[n-k]|^2}{\left|\sum_{k\in I} w[n-k]\right|^2} \frac{N_0}{E_x} = \frac{K}{K^2}\frac{N_0}{E_x}$
- Conclusion: Variance error decreases with kernel width

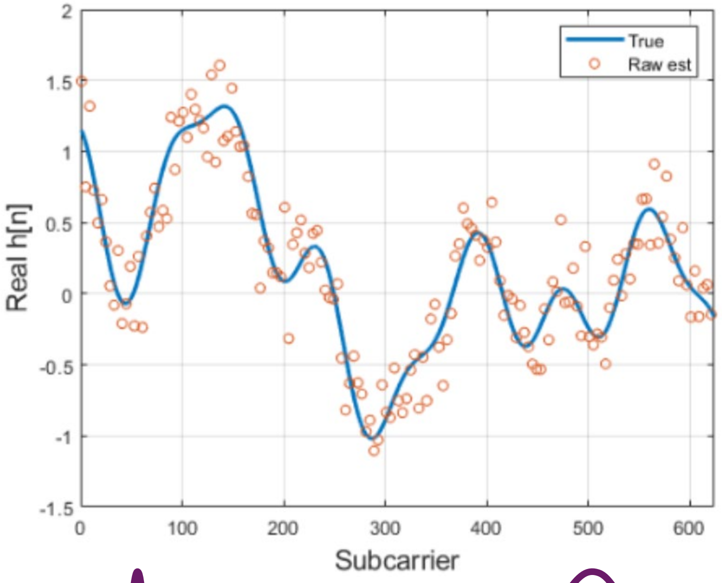
In general, increasing kernel width, decreases variance error

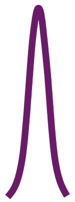


Bias Error

- Signal error:
  - $e_s[n] = h[n] - \frac{1}{z_0[n]} \sum_{k \in I} w[n-k]\, h[k] = h[n] - \bar{h}[n]$
- $\bar{h}[n]$ = local average of $h[k]$
  - Write as: $\bar{h}[n] = \sum_{k \in I} \alpha_{n-k}\, h[k],\ \alpha_{n-k} = \frac{w[n-k]}{\sum_{k \in I} w[n-k]}$
  - Weighted average of values of $h[k]$
  - Weights α_ℓ sum to one
- Bias error, $|e_s[n]|^2$, is large when:
  - Average over region where $h[n]$ changes significantly
  - Window $\geq$ Coherence bandwidth / time

In general, increasing kernel increases the bias error

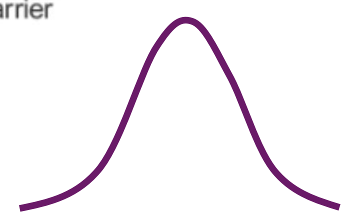




Low bias error

Kernel width

$\ll$ Coherence BW



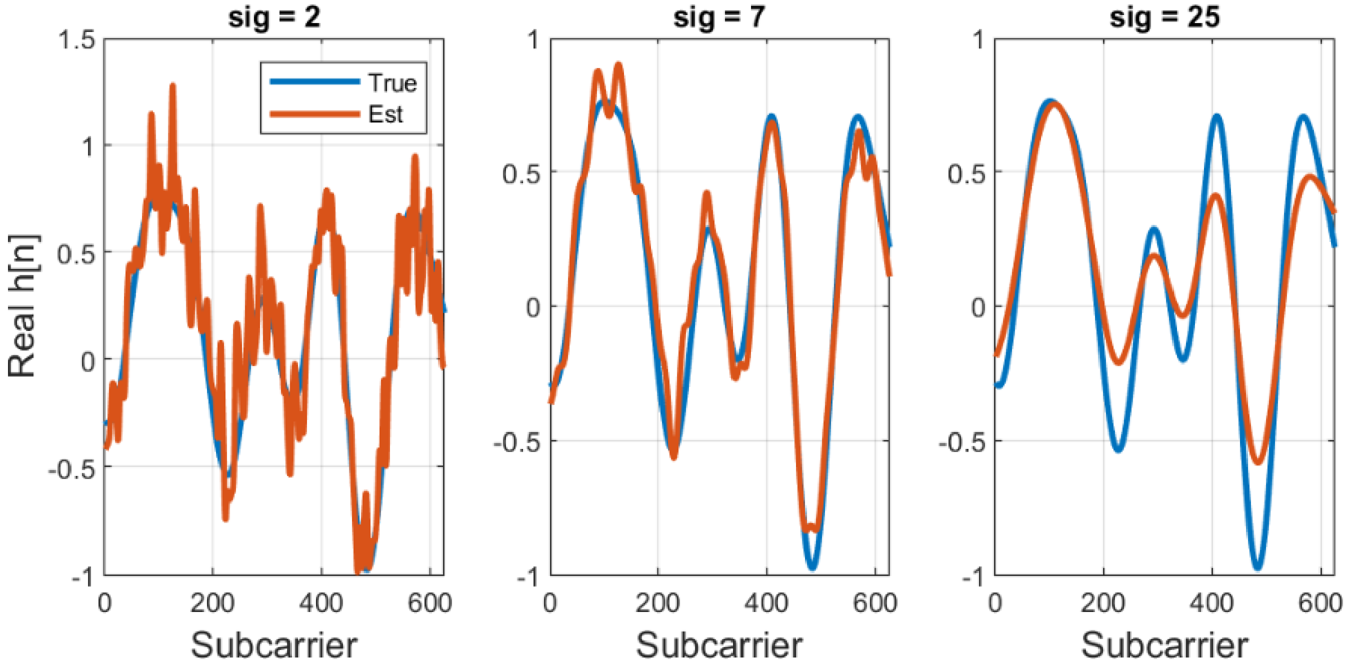
High bias error

Kernel width

$\geq$ Coherence BW

Bias-Variance Tradeoff

Decreasing kernel width



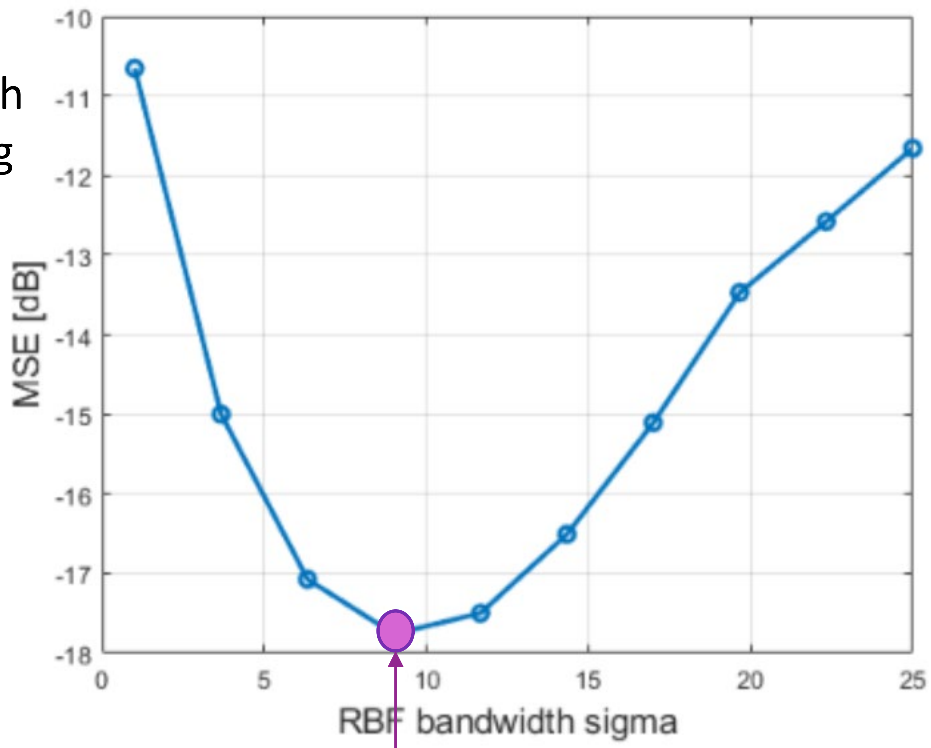
Increasing kernel width

High variance
Too low averaging

High bias
Too much averaging

Bias-Variance Tradeoff

High variance
Small kernel width
Too low averaging



Optimal

High bias
High kernel width
Too much averaging

- This example:
  - Mean delay spread, $\delta = 200$ ns
  - Coherence BW, $W_{coh} = \frac{1}{2\delta} = 2.5$ MHz
  - Optimal width:
    $\sigma = 9$ subcarriers
    $\sigma = 135$ kHz
  - We see: $\sigma \ll W_{coh}$

Modeling Channel Estimation Error

- Channel estimation error: $\hat{h} = h + v$
- Channel estimation error has two components: $E|v|^2 = Bias^2 + Var$
  - Each term depends on level of averaging in the kernel estimator
- Bias error:
  - Typically scales with the channel power
  - Since medium is linear, increasing TX power will not change the relative kernel error
  - Assume $Bias^2 = c_b|h|^2$
  - Constant c_b increases with the kernel width
- Variance error:
  - From earlier, we can write $Var = c_v \frac{N_0}{E_x}, \quad c_v = \frac{\sum_{k\in I} |w[n-k]|^2}{\left|\sum_{k\in I} w[n-k]\right|^2}$
  - Constant c_v decreases with the kernel width

Effective Noise in Equalization

- Consider channel on a single resource element:
  - Channel: $r = hx + w, \quad w \sim CN(0, N_0), \quad E|x|^2 = E_x$
  - SNR is $\gamma_0 = \frac{E_x|h|^2}{N_0}$
- Simple model for channel estimation error: $\hat{h} = h + v, \; v \sim CN(0, N_v)$
  - Bias+ variance channel estimation error: $N_v = c_b|h|^2 + \frac{c_v N_0}{E_x}$
- Consider equalized estimate:

$$z = \hat{h}^* r = (h+v)^*(hx+w) = |h|^2 x + n,$$

  - Total $n = (h+v)^* w + v^* hx$
  - Neglecting cross terms: $n \approx h^* w + v^* hx$
- Total effective noise, n, has two terms:
  - Original noise: $h^* w$
  - Distortion from channel estimation error: $v^* hx$

Signal-to-Noise and Distortion (SNDR)

- Equalized symbol:

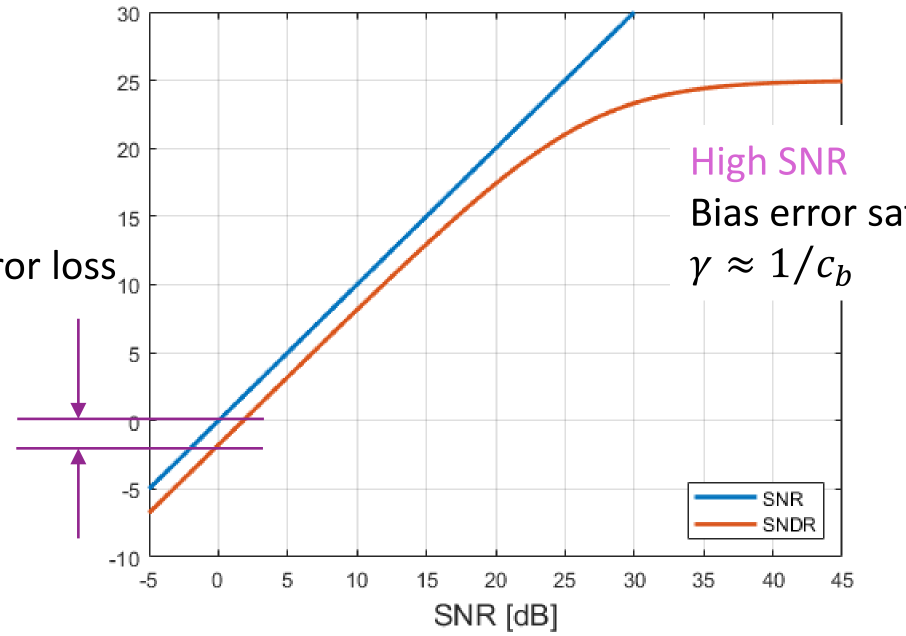
$$z = \hat{h}^* r = (h+v)^*(hx+w) = |h|^2 x + n, \qquad n \approx h^* w + v^* hx$$

- Signal energy: $E_s = |h|^4 E_x$
- Total effective noise energy:
  - Error energy is $N_{tot} = E|n|^2 = |h|^2 N_0 + N_v |h|^2 E_x$
- Signal-to noise and distortion (SNDR): $\boxed{\gamma = \frac{\gamma_0}{1+c_v+c_b\gamma_0}}$
  - Proof: $\gamma = \frac{E_s}{N_{tot}} = \frac{|h|^4 E_x}{|h|^2 N_0 + |h|^2 E_x N_v} = \frac{|h|^2 E_x}{N_0 + N_v E_x}$
  - Substitute in $N_v = \frac{c_v N_0}{E_x} + c_b |h|^2$

SNDR Regimes

- SNDR: $\gamma = \frac{\gamma_0}{1+c_v+c_b\gamma_0}$
- Low SNR regime:
  - $\gamma \approx \frac{\gamma_0}{1+c_v}$
  - Loss in SNR of $\frac{1}{1+c_v}$
  - Limited by variance error, c_v
- High SNR regime:
  - $\gamma_0 \approx \frac{1}{c_b}$
  - Limited by bias error, c_b

Low SNR
Variance error loss
$1/(1+c_v)$



High SNR
Bias error saturation
$\gamma \approx 1/c_b$

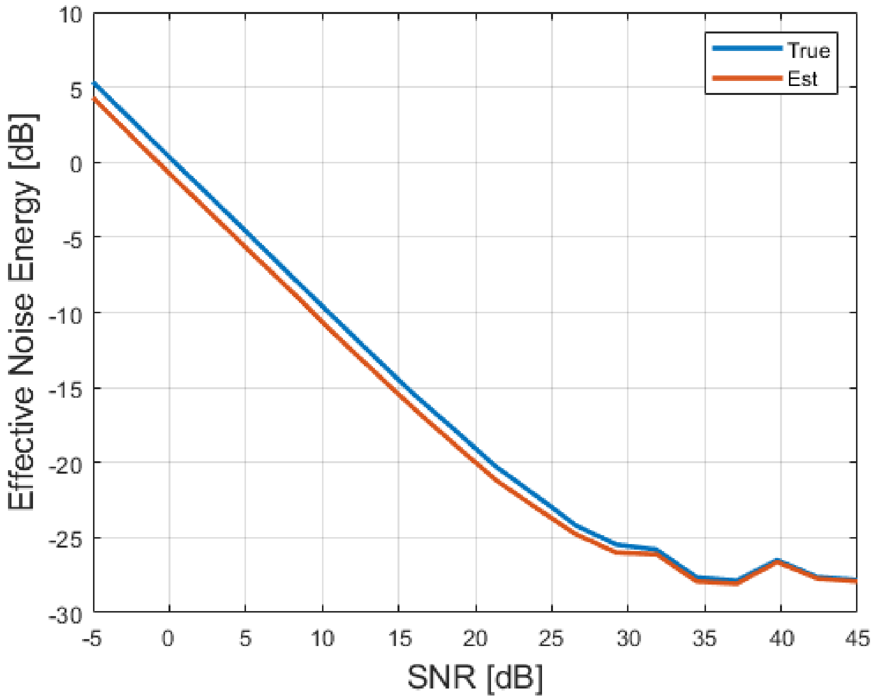
Estimating Effective Noise

- For equalization and LLR calculations we need to estimate the effective noise
- Noise estimation can be performed via residual error:
  - Reference symbols: $r_n = h_n x_n + w_n, \; n = 1, \ldots, N_{RS}$
  - Obtain channel estimate: $\hat{h}_n$ (e.g., via kernel regression)
  - Effective noise estimate: $\widehat{N}_{RS} = \frac{1}{N_{RS}} \sum_n \left| r_n - \hat{h}_n x_n \right|^2$
- Residual error estimate captures noise and distortion
  - If $\hat{h}_n = h_n + v_n$, then $r_n - \hat{h}_n x_n = r_n - h_n x_n + v_n x_n = w_n + v_n x_n$
- But the residual error estimate is biased:
  - Under-estimates the variance error in the distortion
  - Measuring error on the symbols used in training
  - Can compensate by scaling the noise estimate before using in the equalizer or demodulator

Measuring True Effective Noise

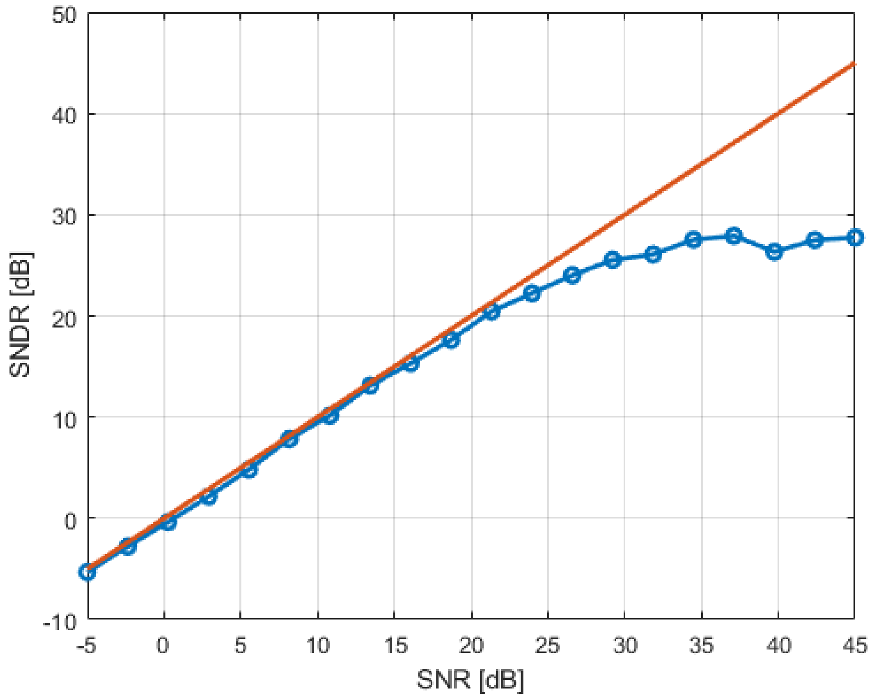
- From previous slide, residual error noise estimate is biased
  - Under-estimates the variance error
  - Uses training symbols for error estimation
- To more accurately measure noise, use separate set of test symbols
  - Get test symbols: $r_n = h_n x_n + w_n, \; n = 1, \ldots, N_{test}$
  - These symbols are distinct from the RS used in training
  - Error estimate on the test: $\widehat{N}_{test} = \frac{1}{N_{test}} \sum_n \left| r_n - \hat{h}_n x_n \right|^2$
- Rarely used during actual operation of the receiver
  - Adding test symbols adds excessive overhead
- But, the test symbols can be done in test and simulation
  - For example, populate data symbols with known symbols

Noise Estimation Simulation



- Simulation parameters as before
  - Mean delay spread, $\delta = 200$ ns
  - Coherence bandwidth, $W_{coh} \approx \frac{1}{2\delta} = 2.5$ MHz
  - One RS per 4 sub-carriers: 60 kHz
- "True" noise measured on data symbols
  - Not used in training
- "Estimated" noise measured on RS symbols
- We see:
  - Estimated noise is slightly optimistic at low SNRs
  - Both true and estimated noise levels off at high SNR
  - Limited by bias SNR

SNDR in Practice



- Estimated the SNDR from effective noise estimate
- In this simulation:
  - About 0.4 dB loss at low SNRs
  - About 27 dB saturation

In-Class Exercise

In-Class Exercise: Degradation in BER due to Channel Estimation

Modify the above program to:

- Populate the reference symbols with QPSK as before
- Populate the data symbols with 16-QAM
- Measure the BER on the data symbols as a function of the SNR for both perfect channel estimation and actual channel estimation.
- Plot the two BER values