

Unit 3. Multi-Path Fading

ECE-GY 6023. WIRELESS COMMUNICATIONS

PROF. SUNDEEP RANGAN



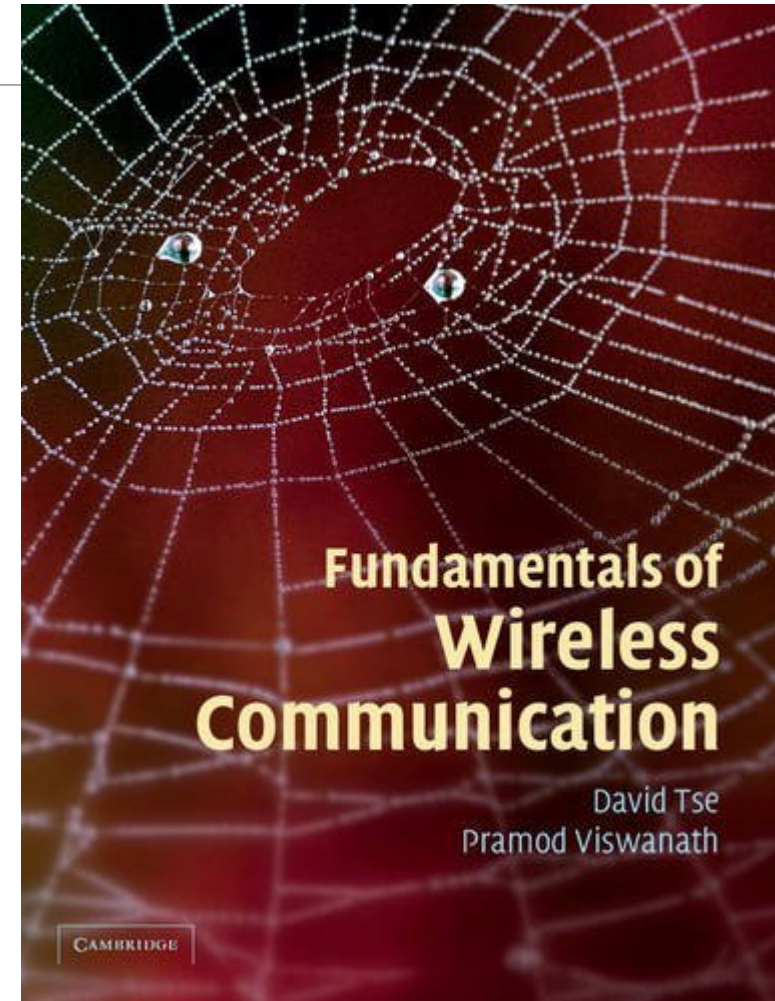
Fading

From the Introduction of a classic text:

There are two fundamental aspects of wireless communication that make the problem challenging and interesting.

*...First is the phenomenon of **fading** ...*

*...Second ...there is significant **interference** ...*



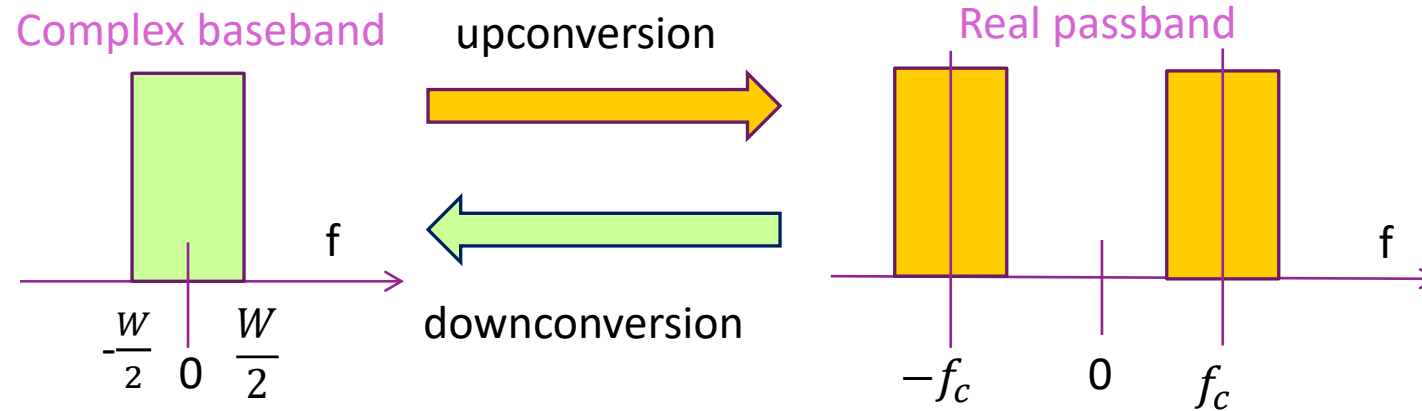
Learning Objectives

- Describe up and down-conversion in time- and frequency-domain
- Describe the steps in the DAC and ADC including the filtering
- Compute a discrete-time and continuous-time base equivalent channels from the passband
- Simulate fractional delays and gains in the sampled data
- Describe and simulate a deterministic multi-path wireless channel
- Compute the time-varying frequency response given the path parameters
- Describe a statistical model for multi-path fading
- Approximately compute the coherence time and bandwidth given a channel

Outline

- ➔ Review of Up- and Downconversion
 - Review of TX and RX Sampling
 - Doppler and Multi-Path Fading
 - Statistical Descriptions of Fading

Up- and Downconversion



RF communication systems:

- Information occurs and is processed in **complex baseband**
- Transmitted and received in **real passband**

Up and down-conversion: Shift center frequency of signals

Also called **mixing**

Up and Down-Conversion in Time Domain

Complex baseband:

- Two real signals, $u_I(t), u_Q(t)$
- Or one complex signal:

$$u(t) = u_I(t) + ju_Q(t)$$

upconversion



$$u_p(t) = \text{Re}(u(t)e^{j\omega_c t})$$

Real passband: $u_p(t)$

downconversion



$$v(t) = 2u_p(t)e^{-j\omega_c t}$$

$$u(t) = h_{LPF}(t) * v(t)$$

Note: downconversion needs:

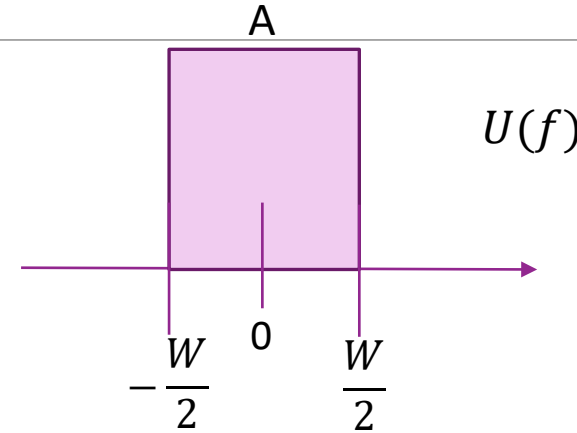
- Multiplication by 2
- Low pass filtering



Mixing in Frequency Domain

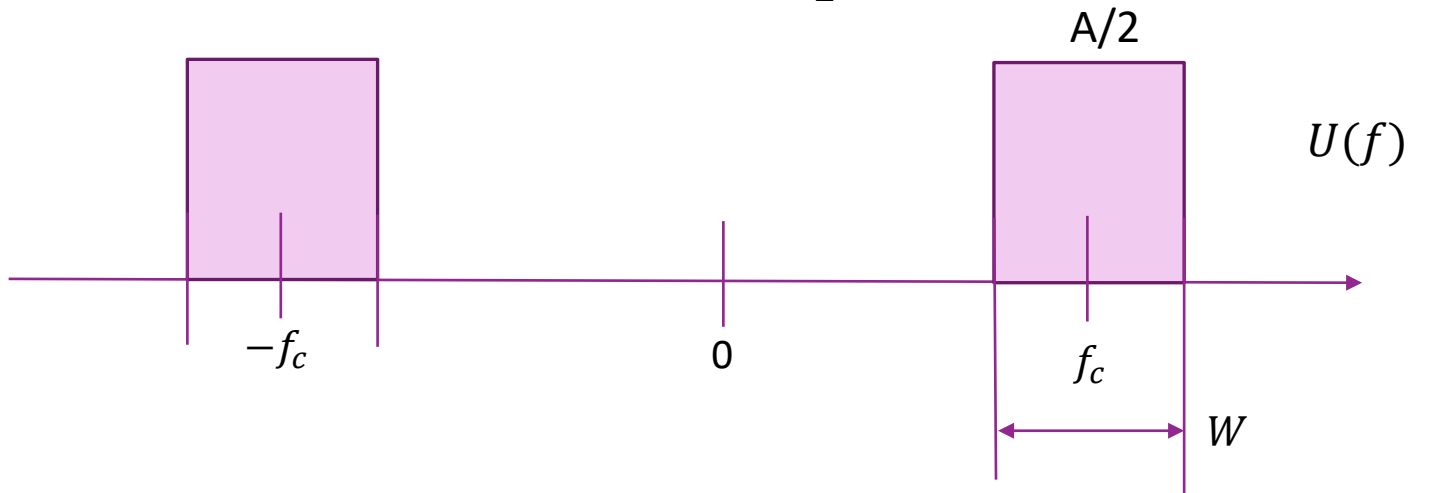
□ Baseband signals

- Centered around $f = 0$, complex
- $\frac{W}{2}$ = single sided bandwidth
- W = two sided bandwidth
- Band-limited to $|f| \leq \frac{W}{2}$



□ Passband signals

- Centered around $f = f_c$, real
- W = bandwidth (per side or image)
- Band-limited to $|f - f_c| \leq \frac{W}{2}$



Discrete IQ Mixer

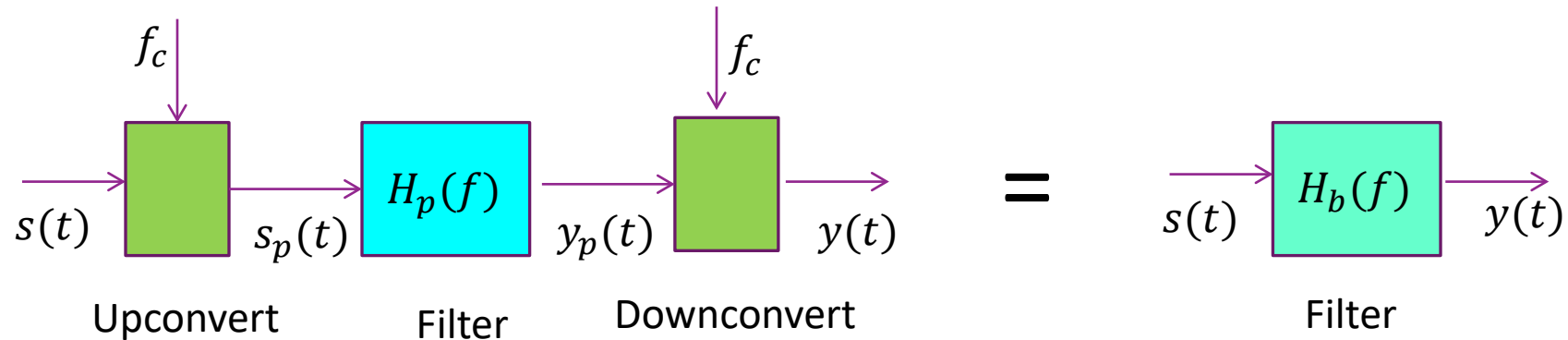


- ❑ LO = “local oscillator” = square or sine wave at f_c
- ❑ I1, I2 = I and Q inputs.
 - Generally, lowpass
- ❑ RF = passband output centered at f_c

http://www.markimicrowave.com/Mixers/IQ_Quadrature-IF_Double-Balanced/IQ-0318.aspx

Datasheet	RF [GHz]	LO [GHz]	IF [MHz]	Conversion Loss [dB]	Image Rejection [dB]	Amplitude Deviation [dB]	Phase Deviation [Degrees]	Isolations L-R [dB]	Isolations L-I [dB]
IQ-0318	3 to 18	3 to 18	DC to 500	7	22	0.75	10	40	20

Baseband Equivalent Channel

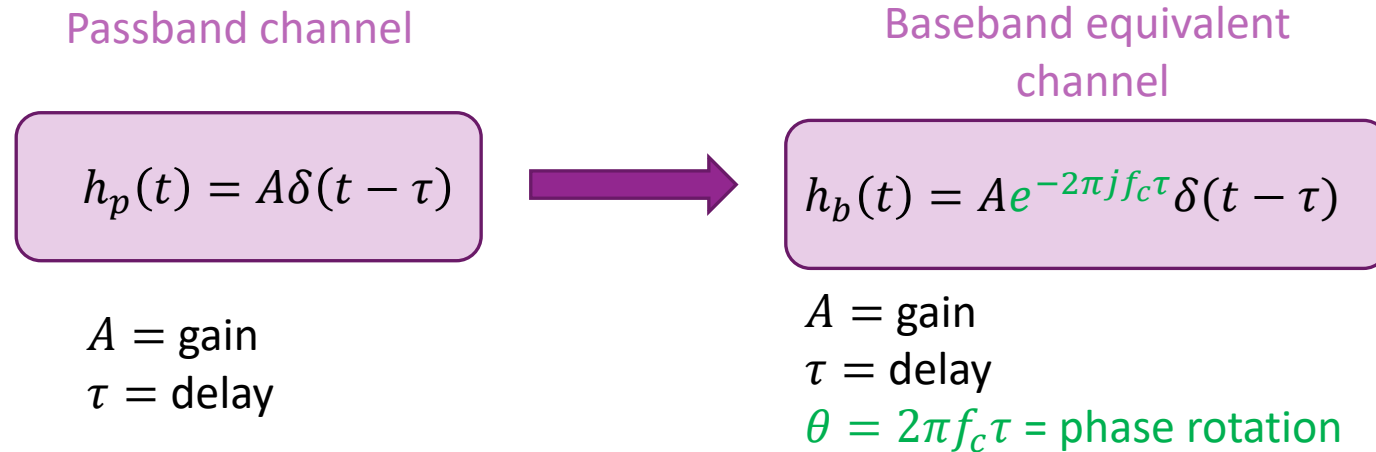


- Filtering at passband equivalent to complex baseband filter
- Assuming downconversion filter is ideal:

$$H_b(f) = H_p(f + f_c) \quad \text{for } |f| \leq \frac{W}{2}$$

- Simply shift $H_p(f)$ to the left by f_c .

Important Special Case: Delay

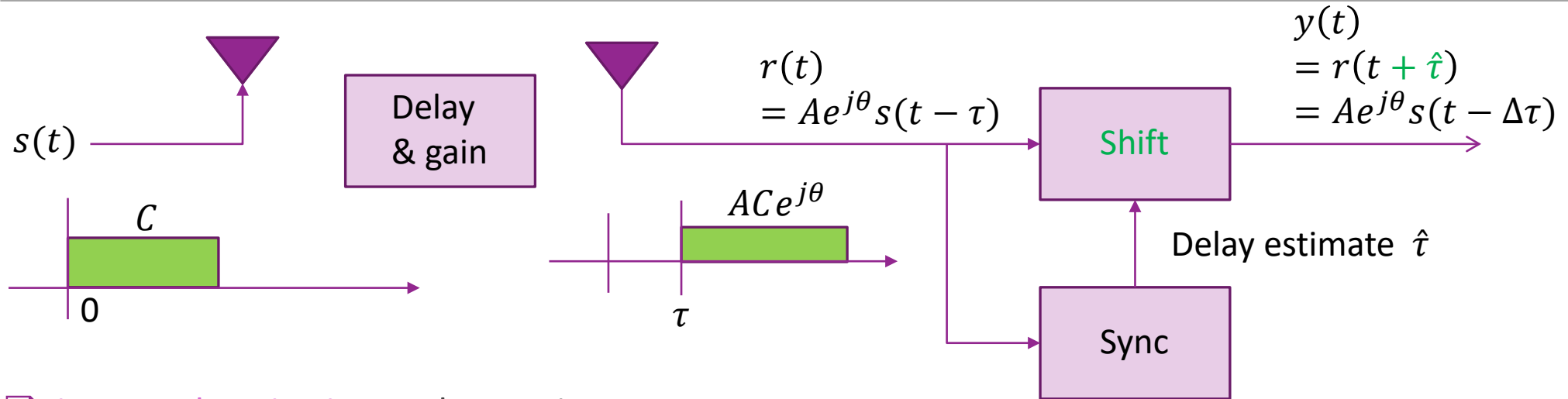


□ Delay, gain in passband \Rightarrow delay, gain and **phase rotation** in baseband

□ Proof:

- Passband frequency response is: $H_p(f) = Ae^{-2\pi j f \tau}$
- Baseband frequency response: $H_b(f) = H_p(f + f_c) = Ae^{-2\pi j (f_c + f) \tau}$
- Equivalent impulse response: $h_b(t) = Ae^{-2\pi j f_c \tau} \delta(t - \tau)$

Synchronization and Delay Errors



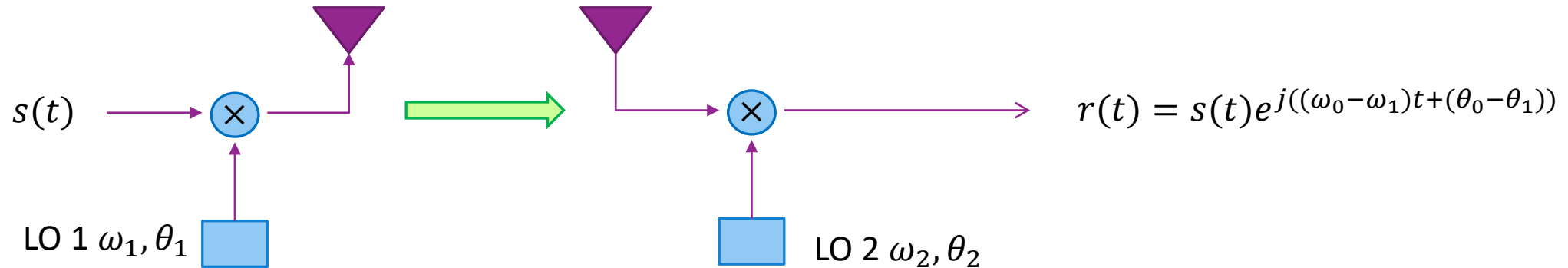
Time synchronization at the receiver:

- Estimate the arrival time of the signal $\hat{\tau}$
- Starts processing remainder of signal starting at $\hat{\tau}$
- Equivalent to shifting received signal ahead in time by $\hat{\tau}$: $y(t) = r(t + \hat{\tau})$
- Remaining time error: $\Delta\tau = \tau - \hat{\tau}$

Later, we will discuss:

- How to estimate τ (synchronization) and how to correct for gain and phase error (equalization)

Frequency Errors



❑ Oscillators at TX and RX always have some mismatch. To analyze, suppose:

- Upconversion: $s_p(t) = \text{Re}(s(t)e^{j\omega_1 t + \theta_1})$,
- Downconversion: $r(t) = \text{LPF}(2s_p(t)e^{-j(\omega_2 t + \theta_2)})$

❑ LO error leads to time-varying gain: $r(t) = g(t)s(t)$, $g(t) = e^{j((\omega_0 - \omega_1)t + (\theta_0 - \theta_1))}$

- Frequency and phase shift

In-Class Problems

Problem 1: Plotting a Frequency Response of a Delay Channel

Consider a system with the following parameters

```
bw = 20e6;    % bandwidth
pl = 80;      % path loss (dB)
tau = 200e-9; % timing error
```

Plot the real component of the frequency response at 1024 points over the bandwidth. Assume phase = 0 at DC. Use 1024 frequency points

Problem 2: Plotting the Channel Response from Frequency Error

Suppose a link has the following parameters:

```
fc = 37e9; % carrier freq
loppm = 1; % LO error in ppm
```

Plot the relative change of the gain:

$$E(t) = |g(t) - g(t + \tau)|^2 / |g(t)|^2$$

as a function of tau from 0 to 5 us.

Outline

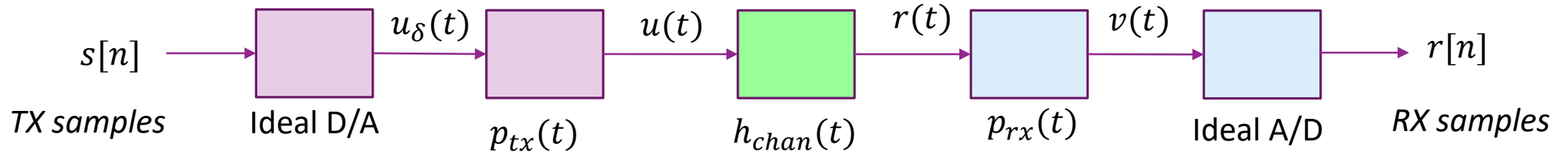
Review of Up- and Downconversion

 Review of TX and RX Sampling

Doppler and Multi-Path Fading

Statistical Descriptions of Fading

Typical Digital Communication Path



- ❑ All modern communication systems TX and RX **digital samples**
- ❑ **Transmitter**: DAC + filtering with $p_{tx}(t)$. Filtering used to:
 - Suppress out of band emissions
- ❑ **Receiver**: Filters with $p_{rx}(t)$ then performs ADC. Filtering plays two roles:
 - Reduces noise
 - Remove out-of-band signals before ADC. (i.e. Anti-aliasing)
- ❑ Filter design discussed in digital communications class

Review of DTFT

- ❑ Given discrete-time sequence $s[n]$
 - Real or complex
- ❑ Discrete-time Fourier Transform: $S(\Omega) = \sum_n s[n]e^{-j\Omega n}$
- ❑ Inverse DTFT: $s[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\Omega)e^{j\Omega n} d\Omega$
- ❑ Note $S(\Omega)$ is always a 2π periodic signal
 - Can take integral for inverse DTFT on any period of 2π
- ❑ Ω is the **discrete frequency**. Units is radians per sample.
- ❑ For finite length signals and finite number of Ω , can be computed via FFT
- ❑ Review in your signals and systems class



Common DTFT Pairs

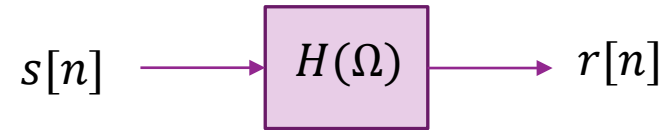
Time domain $x[n]$	Frequency domain $X_{2\pi}(\omega)$
$\delta[n]$	$X_{2\pi}(\omega) = 1$
$\delta[n - M]$	$X_{2\pi}(\omega) = e^{-i\omega M}$
$\sum_{m=-\infty}^{\infty} \delta[n - Mm]$	$X_{2\pi}(\omega) = \sum_{m=-\infty}^{\infty} e^{-i\omega Mm} = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{M}\right)$ $X_o(\omega) = \frac{2\pi}{M} \sum_{k=-(M-1)/2}^{(M-1)/2} \delta\left(\omega - \frac{2\pi k}{M}\right) \quad \text{odd } M$ $X_o(\omega) = \frac{2\pi}{M} \sum_{k=-M/2+1}^{M/2} \delta\left(\omega - \frac{2\pi k}{M}\right) \quad \text{even } M$
$u[n]$	$X_{2\pi}(\omega) = \frac{1}{1 - e^{-i\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ $X_o(\omega) = \frac{1}{1 - e^{-i\omega}} + \pi \cdot \delta(\omega)$
$a^n u[n]$	$X_{2\pi}(\omega) = \frac{1}{1 - ae^{-i\omega}}$
$e^{-i\omega n}$	$X_o(\omega) = 2\pi \cdot \delta(\omega + a), \quad -\pi < a < \pi$ $X_{2\pi}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega + a - 2\pi k)$

See Wikipedia

$\cos(a \cdot n)$	$X_o(\omega) = \pi [\delta(\omega - a) + \delta(\omega + a)],$ $X_{2\pi}(\omega) \triangleq \sum_{k=-\infty}^{\infty} X_o(\omega - 2\pi k)$
$\sin(a \cdot n)$	$X_o(\omega) = \frac{\pi}{i} [\delta(\omega - a) - \delta(\omega + a)]$
$\text{rect}\left[\frac{n - M/2}{M}\right]$	$X_o(\omega) = \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-\frac{i\omega M}{2}}$
$\text{sinc}(W(n+a))$	$X_o(\omega) = \frac{1}{W} \text{rect}\left(\frac{\omega}{2\pi W}\right) e^{i\omega a}$
$\text{sinc}^2(Wn)$	$X_o(\omega) = \frac{1}{W} \text{tri}\left(\frac{\omega}{2\pi W}\right)$



Discrete-Time Systems



□ Consider discrete-time LTI system

□ Time-domain: Characterized by **impulse response** $h[n]$

$$r[n] = h[n] * s[n] = \sum_k h[k]s[n - k]$$

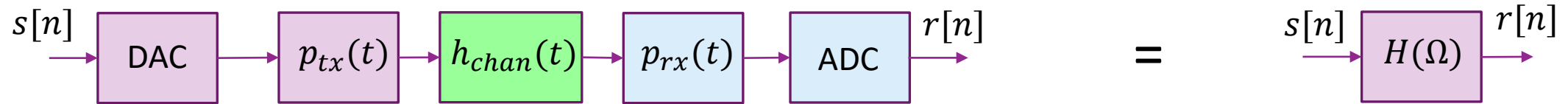
□ Frequency-domain: Characterized by **frequency response** $H(\Omega)$

$$R(\Omega) = H(\Omega)S(\Omega)$$

- $R(\Omega) = \sum r[n]e^{-j\Omega n}$, $r[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(\Omega)e^{j\Omega n} d\Omega$



DT Equivalent Channel



□ Discrete-time baseband equivalent channel:

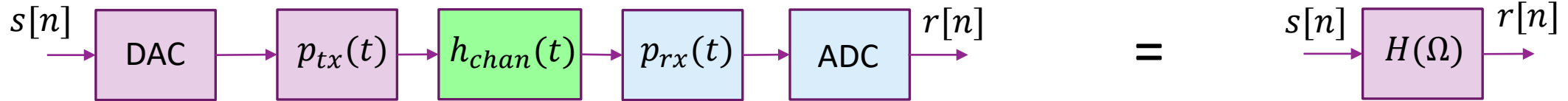
- Describes equivalent mapping from $s[n]$ to $r[n]$
- Includes effects of TX and RX filtering and continuous-time baseband channel

□ Band-limited filters:

- Suppose one of P_{rx}, P_{tx} is bandlimited to $|f| < \frac{1}{2T}$ (no out-of-band emissions or aliasing)
- Then, discrete-time equivalent channel reduces to:

$$H(\Omega) = \frac{1}{T} P_{rx} \left(\frac{\Omega}{2\pi T} \right) P_{tx} \left(\frac{\Omega}{2\pi T} \right) H_{chan} \left(\frac{\Omega}{2\pi T} \right) \quad \text{for } |\Omega| < \pi$$

Ideal Filtering



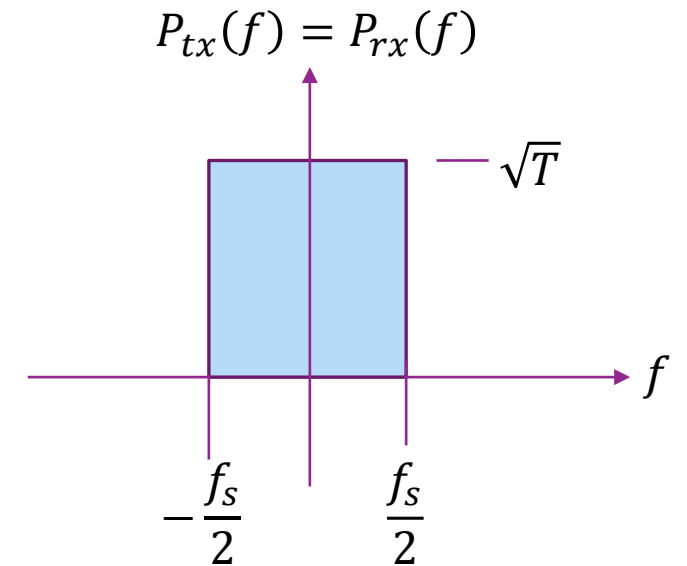
□ Suppose sample rate $f_s = \frac{1}{T}$

□ “Ideal” TX and RX filter :

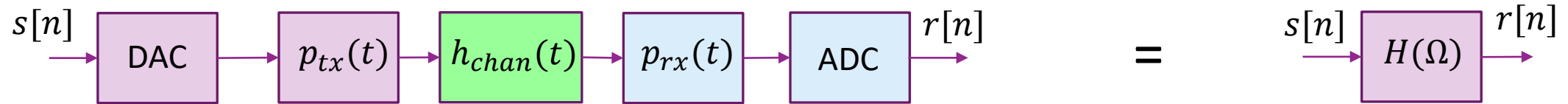
- $p_{tx}(t) = p_{rx}(t) = \frac{1}{\sqrt{T}} \text{Sinc}\left(\frac{t}{T}\right)$
- In frequency domain: $P_{rx}(f) = P_{tx}(f) = \sqrt{T} \text{Rect}(fT)$
- Also called “brick wall” filter

□ Most practical filters match this well

- Up to gain and delay



Ideal Filtering



□ Assume TX and RX filters are ideal

□ **Theorem:** DT equivalent channel is the re-scaled continuous-time channel

$$H(\Omega) = H_{chan}\left(\frac{\Omega}{2\pi T}\right)$$

- Frequency f mapped to $\Omega = 2\pi T f$



Special Case: Delay

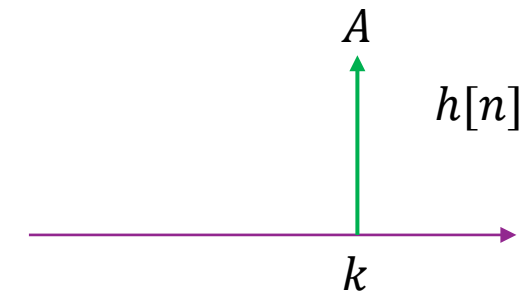
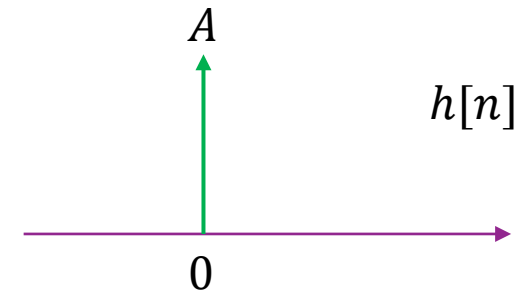
	Passband	Continuous-Time Baseband	Discrete-Time Baseband
Impulse response	$h_p(t) = A\delta(t - \tau)$	$h_{chan}(t) = Ae^{-j\omega_c\tau}\delta(t - \tau)$	$h[n] = Ae^{-j\omega_c\tau} \text{sinc}\left(\frac{\tau n}{T}\right)$
Frequency response	$H_p(f) = Ae^{-j2\pi f\tau}$	$H_{chan}(f) = Ae^{-j\omega_c\tau} e^{-j2\pi f\tau}$	$H(\Omega) = Ae^{-j\omega_c\tau} e^{-j\Omega\tau/T}$

- Suppose passband has a gain and delay.
- Then discrete-time frequency-domain: gain and linear phase rotation over frequency
 - Rotates $2\pi \tau/T$ radians every period
- In discrete-time time-domain: gain, constant phase rotation and sinc filter with delay



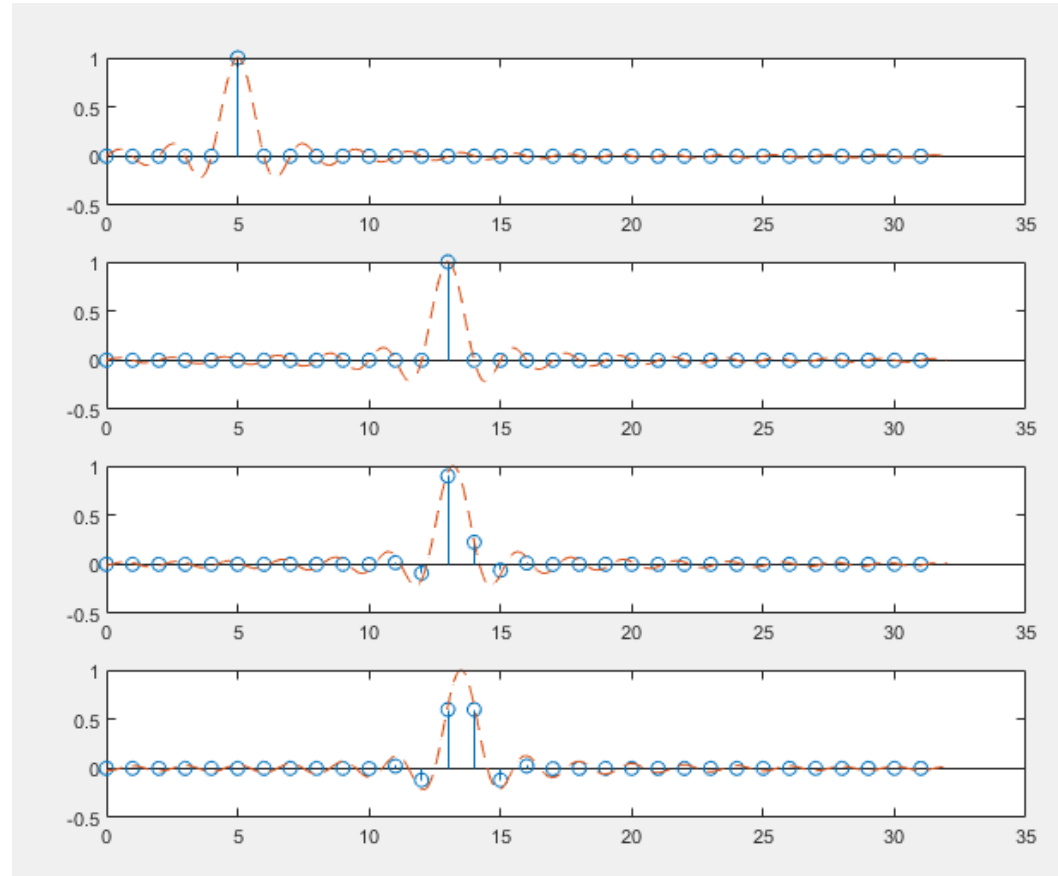
Sinc Filter with Integer Delays

- Suppose we have ideal filtering and passband has delay and gain
- From previous slide, $r[n] = h[n] * s[n]$, $h[n] = Ae^{-j\omega_c\tau} \text{sinc}\left(\frac{\tau n}{T}\right)$
- Special case 1: **No delay** $\tau = 0$:
 - $h[n] = A\delta[n] \Rightarrow r[n] = As[n]$
 - Baseband channel introduces only gain
- Special case 2: **Integer delays** $\tau = kT$:
 - $h[n] = A\delta[n - k] \Rightarrow r[n] = As[n - k]$
 - Baseband channel introduces gain and integer shift
- Ex: Suppose sample rate is 20 MHz and signal is delayed by 400 ns.
 - Integer delay in discrete-time signal is $\frac{20}{0.4} = 50$ samples



Sinc Pulses with Fractional Delay

- $h[n] = Ae^{-j\omega_c\tau} \text{sinc}\left(\frac{\tau n}{T}\right)$
- Causes blurring over multiple samples
- Inter-symbol interference
- Will need equalization to correct
 - More on this later



$\tau = 5$

$\tau = 13$

$\tau = 13.2$

$\tau = 13.5$

Simulating Fractional Delays in MATLAB

- ❑ Code on previous slide was create with DSP toolbox

```
tau = [0,8,8.2,8.5]; % Delays in fractions of a sample
```

```
% Create a fractional delay object from the DSP toolbox  
% We select the Farrow interpolation, which is a fast  
% and accurate method. It is important to select the options  
% correctly  
dly = dsp.VariableFractionalDelay(...  
    'InterpolationMethod', 'Farrow', 'FilterLength', 8, ...  
    'FarrowSmallDelayAction', 'Use off-centered kernel');
```

```
% Create delays of the sequence  
y = dly.step(x, tau);
```

Creates T x D matrix
Row i is delayed by $\tau(i)$

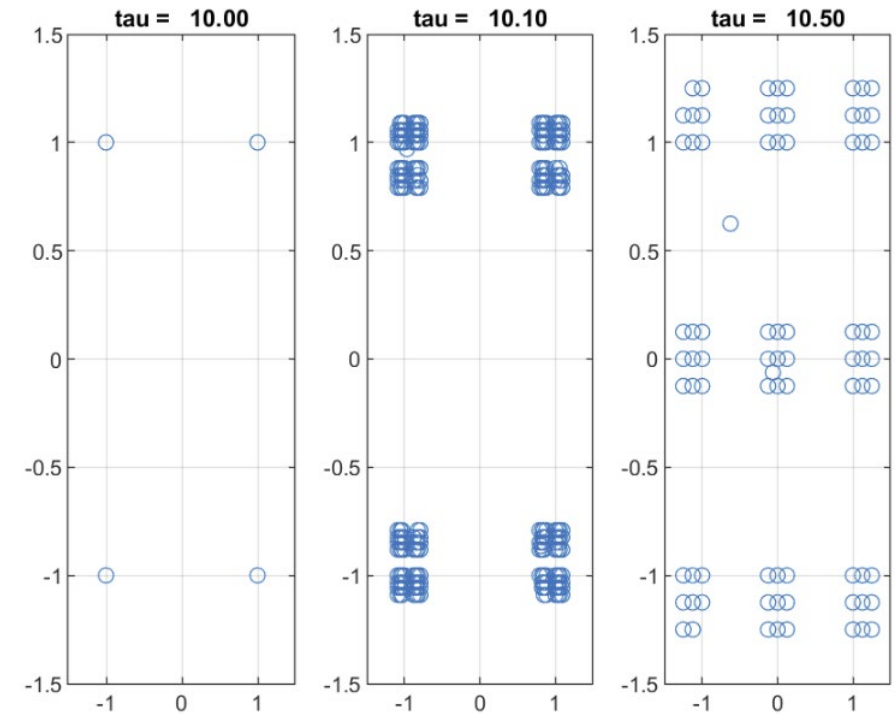
In-Class Problem: Fractional Delays on Constellations

Problem 3: Effect of Delay on a QAM Constellation

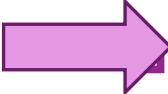
In this problem, we will show the effect of fractional delays on a TX constellation.

First, we generate random QPSK symbols.

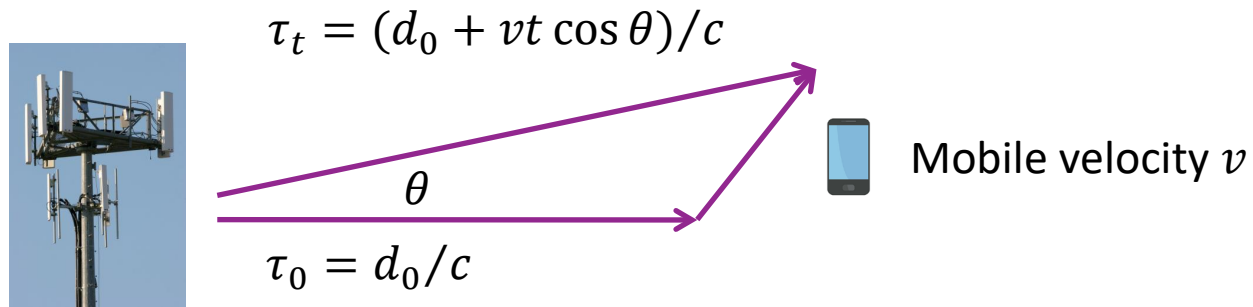
```
% TODO: Generate nb=1024 bits using the randi command.  
% bits = ...  
nb = 1024;  
bits = randi(2,nb,1)-1;
```



Outline

- Review of Up- and Downconversion
- Review of TX and RX Sampling
-  Doppler and Multi-Path Fading
- Statistical Descriptions of Fading

Receiver with Local Motion



□ With the RX in motion, the propagation delay changes with time.

□ In complex baseband signal:

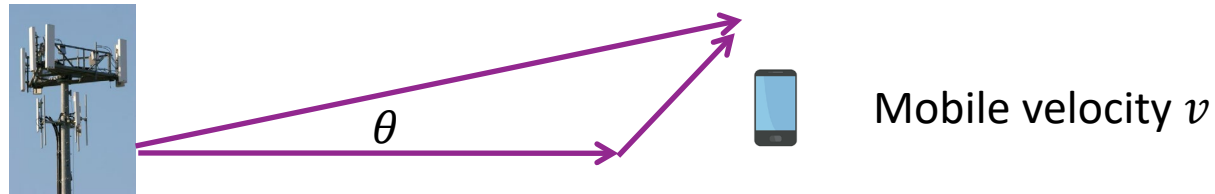
$$r(t) = \alpha e^{-j\omega_c \tau(t)} u(t - \tau_t) = \alpha e^{j2\pi \left(-\frac{tv}{c} \cos \theta - d_0 f_c \right)} u(t - \tau_t)$$

□ **Local motion assumption:** $u(t - \tau_t) \approx u(t - \tau_0)$ for t small

- Effect of the change in propagation delay is only in the complex exponential

□ Then: $r(t) = \alpha e^{j2\pi \left(-\frac{tv}{c} \cos \theta - d_0 f_c \right)} u(t - \tau_0)$

Doppler Shift



□ Single path with local motion:

$$r(t) = g_0 e^{j2\pi f_d t} u(t - \tau_0)$$

- Complex gain: $g_0 = \alpha e^{-2\pi j \tau_0 f_c}$
- Doppler shift: $f_d = -\frac{v f_c \cos \theta}{c}$
- Delay: τ_0

□ For a single path: Local motion causes a phase rotation, but no change in amplitude

Example: Computing Doppler Shift



- Suppose: carrier frequency is $f_c = 2.1 \text{ GHz}$ Car moves towards a base station at 100 km/h .
- What is the Doppler shift?
- Answer: $v = 100 \text{ km/h} = 27.7 \text{ m/s}$, $c = 3(10)^8 \text{ m/s}$, $\theta = 180$:

$$\Delta f = -\frac{vf_c \cos \theta}{c} = -\frac{(27.7)(2.1)(10)^9(-1)}{3(10)^8} \approx 194 \text{ Hz}$$

- If the angle away from BS at $\theta = 45$:

$$\Delta f = \frac{vf_c \cos \theta}{c} = \frac{(27.7)(2.1)(10)^9 \cos(45)}{3(10)^8} \approx -138 \text{ Hz}$$

Multi-Path Models

- Most channel consists of many paths
 - Direct paths
 - Reflections, transmissions, diffraction, ...
 - LOS and NLOS paths

- Wideband time-domain baseband model:

$$r(t) = \sum_{\ell=1}^L g_{\ell} e^{j\omega_{\ell} t} x(t - \tau_{\ell})$$

- g_{ℓ} : Complex path gain
- $\omega_{\ell} = -\frac{2\pi v f_c}{c} \cos \theta_{\ell}$: Doppler shift of path
- τ_{ℓ} : Delay of the path

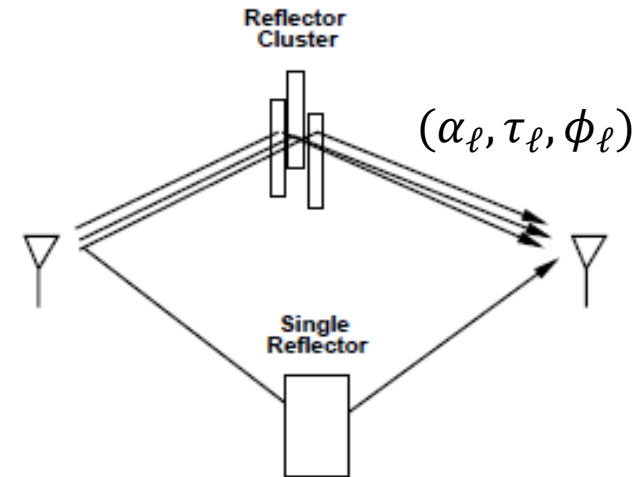


Figure 3.1: A Single Reflector and A Reflector Cluster.

Time-Varying Frequency Response

□ Multipath channel: $y(t) = \sum_{\ell=1}^L g_{\ell} e^{j\omega_{\ell} t} x(t - \tau_{\ell})$

□ Consider exponential input: $x(t) = e^{j\omega t}$

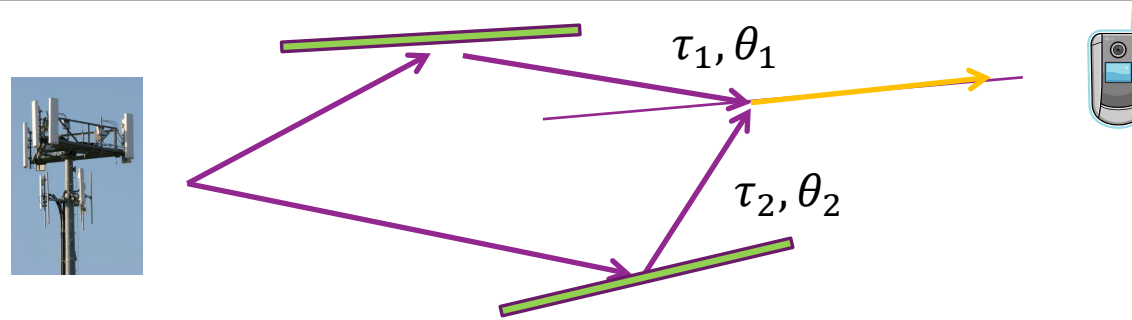
□ Output is: $y(t) = H(t, \omega)x(t)$

□ Time-varying frequency response

$$H(t, \omega) = \sum_{\ell=1}^L g_{\ell} e^{j(\omega_{\ell} t - \omega \tau_{\ell})}$$

□ May also write: $H(t, f) = H(t, 2\pi f)$

Example with Two Paths



- To simplify understanding, consider two path model

$$r(t) = h_1 e^{j\omega_1 t} u(t - \tau_1) + h_2 e^{j\omega_2 t} u(t - \tau_2)$$

$$\omega_i = -2 \pi f_{dmax} \cos \theta_i$$

- Time-varying response:

$$H(t, \omega) = h_1 e^{j(\omega_1 t - \omega \tau_1)} + h_2 e^{j(\omega_2 t - \omega \tau_2)}$$

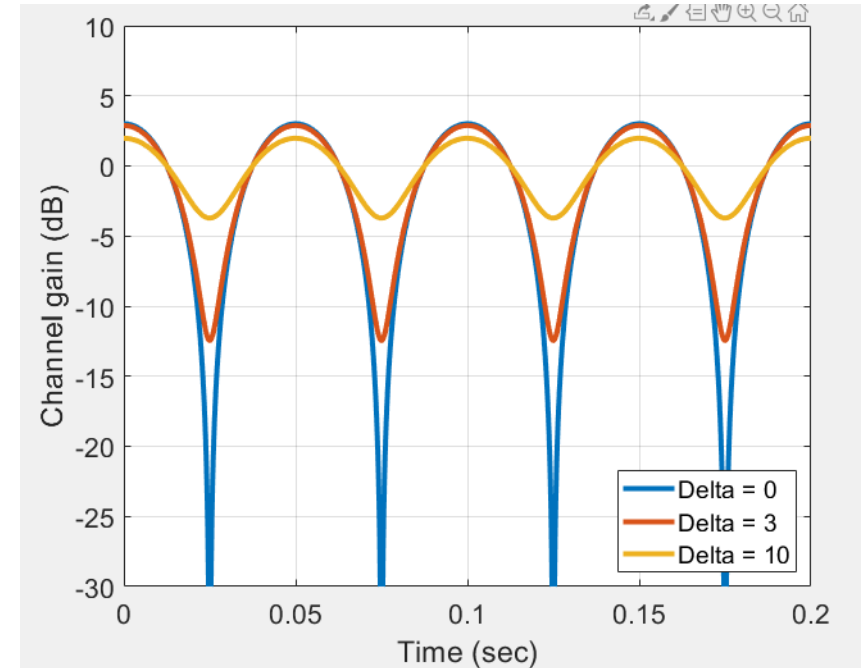
- Power gain:

$$P(t, \omega) = |H(t, \omega)|^2 = |h_1 e^{j(\omega_1 t - \omega \tau_1)} + h_2 e^{j(\omega_2 t - \omega \tau_2)}|^2$$

Variation in Time

- Fixed frequency ω_0
- Look at time variations $P(t, \omega_0)$
- Rate of variation depends on Doppler spread:
$$\Delta f = f_{max}(\cos \theta_1 - \cos \theta_2)$$
- Size of variation depends on spread of gains:
 - Avg: $|h_1|^2 + |h_2|^2$
 - Min: $(|h_1| - |h_2|)^2$: Destructive interference
 - Max: $(|h_1| + |h_2|)^2$: Constructive interference
- With equal path gains, there are nulls

$$P(t, \omega_0) = |h_1 e^{j(\omega_1 t + \phi_1)} + h_2 e^{j(\omega_2 t + \phi_2)}|^2$$



Plot shows $f_{max} = 10$ Hz,
 $\theta_1 = 0, \theta_2 = 180$,
 $h_2 = 10^{-0.05\Delta} h_1, |h_1|^2 + |h_2|^2 = 1$

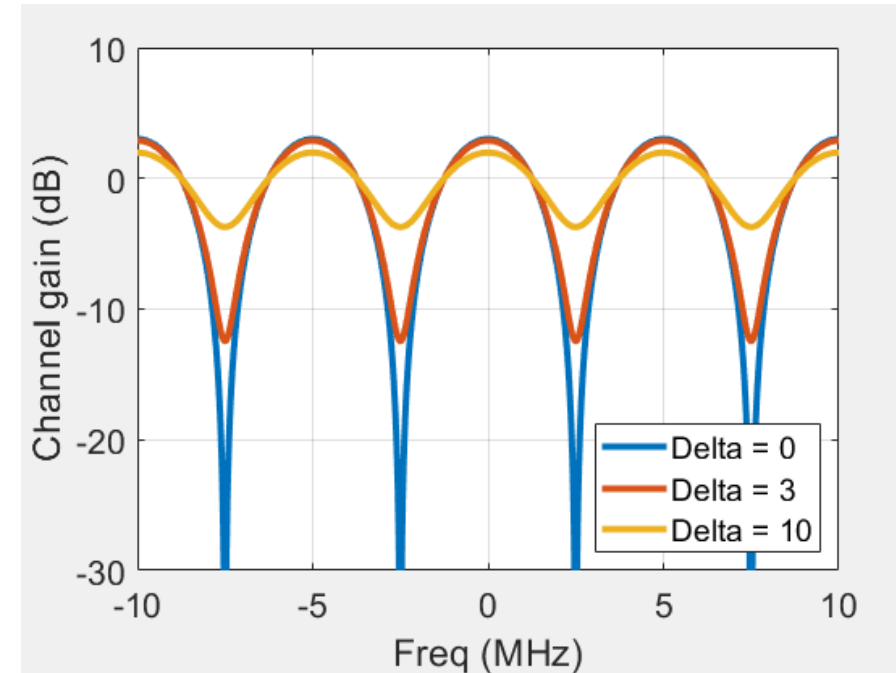
Variation in Frequency

$$P(t_0, \omega) = |h_1 e^{j(\omega\tau_1 + \phi_1)} + h_2 e^{j(\omega\tau_2 + \phi_2)}|^2$$

- Fixed frequency t_0
- Look at time variations $P(t, \omega_0)$
- **Period** of variation depends on **delay spread**:

$$\Delta f = \frac{1}{\tau_2 - \tau_1}$$

- **Size** of variation depends on spread of gains:
 - Avg: $|h_1|^2 + |h_2|^2$
 - Min: $(|h_1| - |h_2|)^2$
 - Max: $(|h_1| + |h_2|)^2$



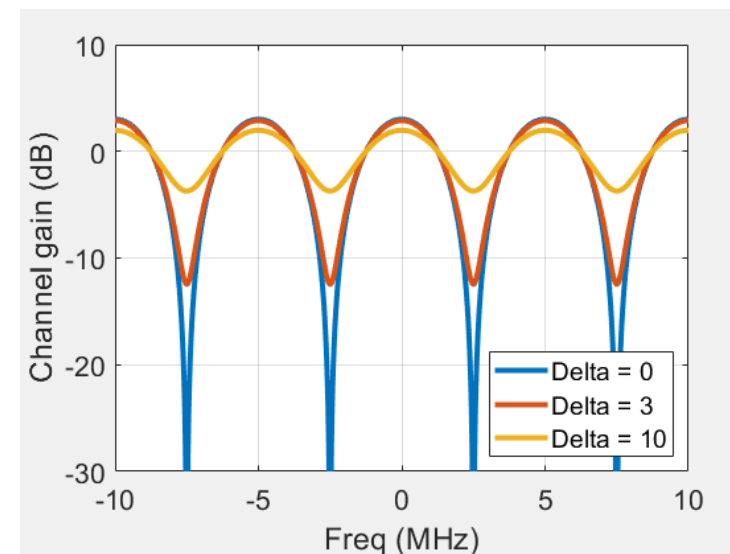
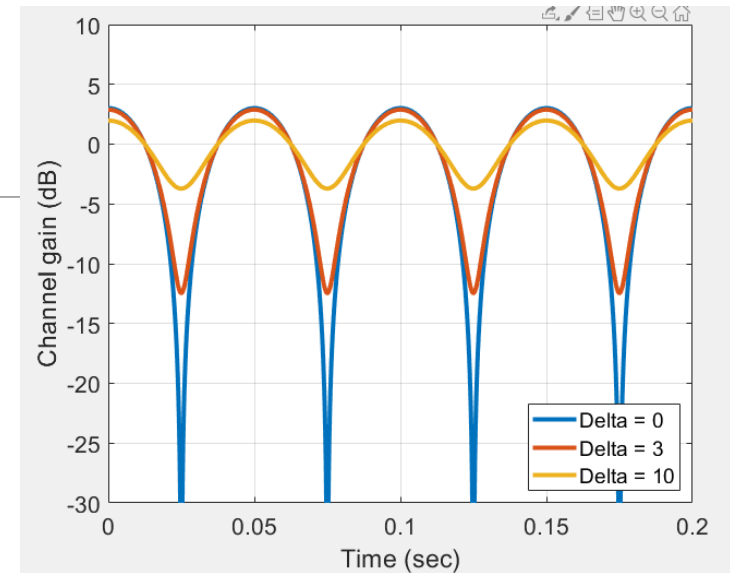
Plot shows

$$\tau_1 = 0, \tau_2 = 200 \text{ ns},$$

$$h_2 = 10^{-0.05\Delta} h_1, |h_1|^2 + |h_2|^2 = 1$$

Fading

- Over time and frequency, paths can either
 - Constructively interfere \Rightarrow Peaks
 - Destructively interfere \Rightarrow Nulls
- Process is called **fading**
 - Intermittent channel quality
- One of the most significant challenges in wireless
- Later, we will discuss how to overcome fading



Narrowband Approximation

Multi-path channel: $r(t) = \sum_{\ell=1}^L g_{\ell} e^{j\omega_{\ell} t} x(t - \tau_{\ell})$

Define **delay spread**: $\delta = \max_{\ell} |\tau_{\ell} - \tau_0|$

- Max path difference in seconds

Narrowband approximation: $X(f)$ is band-limited to $|f - f^0| \ll \frac{1}{2\delta}$ then

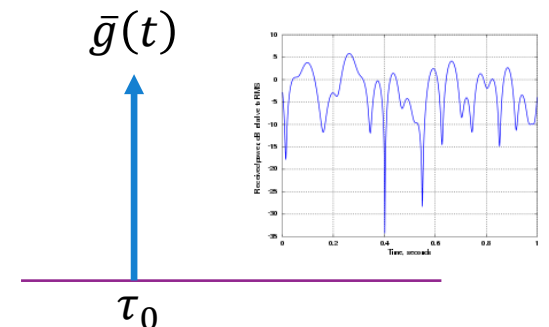
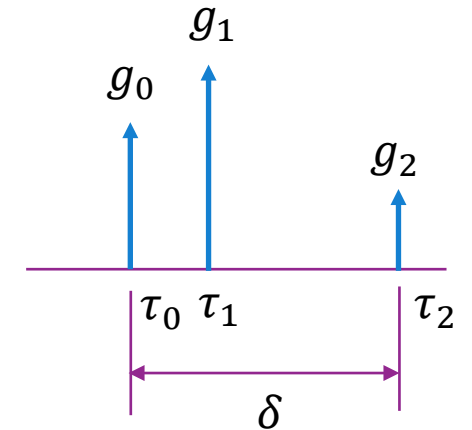
$$r(t) \approx H(t, f^0) x(t - \tau_0)$$

- Proof below

Coherence bandwidth $= \frac{1}{2\delta}$

Effective single path gain: $\bar{g}(t) = H(t, f^0) = \sum_{\ell=1}^L g_{\ell} e^{j\omega_{\ell} t - j\omega^0 \tau_{\ell}}$

- Channel appears as a single path channel with time-varying gain
- Channel gain $\bar{g}(t)$ is band-limited to max Doppler $\max_{\ell} \omega_{\ell}$



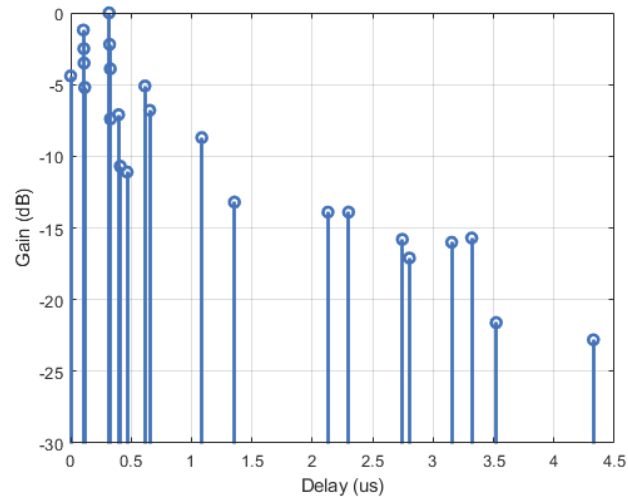
Example: 3GPP Cluster Delay Line Model

```
fc = 2.3e9;           % carrier in Hz
dlySpread = 0.5e-6;   % delay spread in seconds

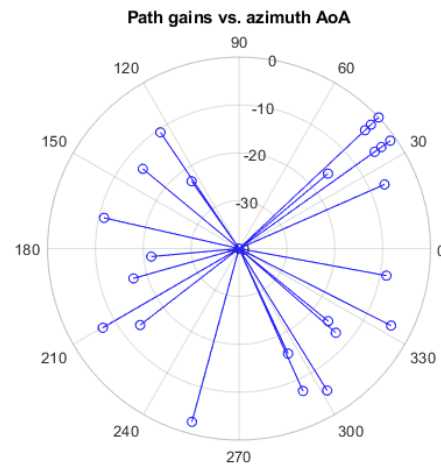
% Get NR channel object and data
chan = nrCDLChannel('DelayProfile','CDL-C',...
    'DelaySpread',dlySpread, 'CarrierFrequency', fc, ...
    'NormalizePathGains', true);
chaninfo = info(chan);
```

- ❑ 3GPP has several deterministic multi-path models
- ❑ Called Cluster-Delay Line (CDL)
- ❑ Can be downloaded in MATLAB 5G Toolbox
 - Gives the gain, delay and angles of each path
 - This ex: CDL-C with 24 paths

Path delay profile



Path AoA azimuth profile



```
gain = chaninfo.AveragePathGains';
aoaAz = chaninfo.AnglesAoA';
aoaEl = 90 - chaninfo.AnglesZoA';
dly = chaninfo.PathDelays';
```



Computing the Doppler of Each Path

□ Computing the Doppler shift of each path

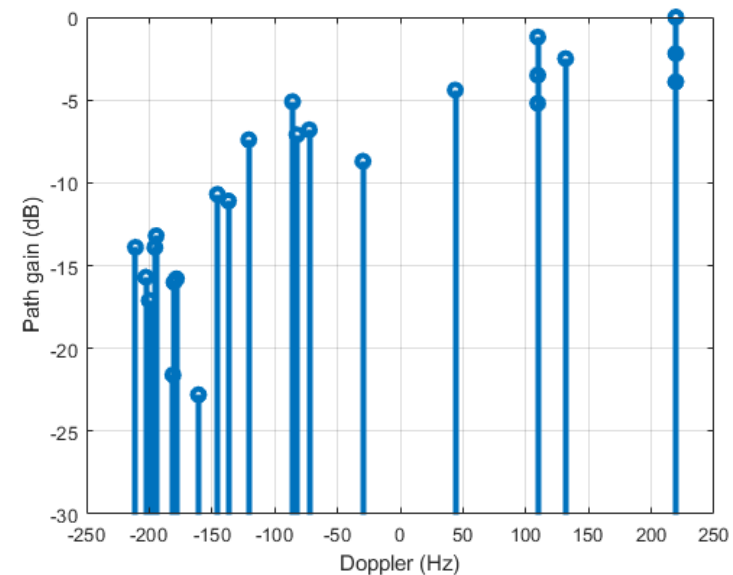
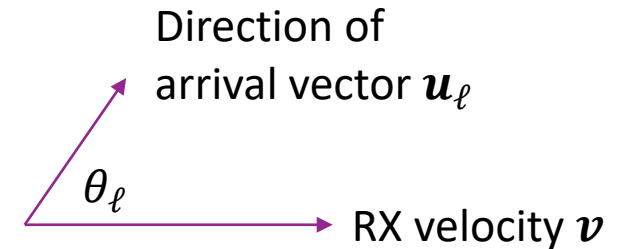
- Suppose that RX has velocity vector $\mathbf{v} = (v_x, v_y, v_z)$
- Doppler shift of path ℓ is: $f_\ell = -\frac{f_c}{c} \mathbf{v}^T \mathbf{u}_\ell = -\frac{f_c}{c} \|\mathbf{v}\| \cos \theta_\ell$

□ In this simulation: $v = 30$ m/s in x-axis

```
% UE velocity vector in m/s
ueVel = [30; 0; 0];

% Create a unit vectors in the direction of the path
[ux, uy, uz] = sph2cart(deg2rad(aoaAz), deg2rad(aoaEl), 1);
U = [ux uy uz];

% Compute Doppler shift
vc = physconst('Lightspeed');
fd = -fc/vc*U*ueVel;
```



Computing the Narrowband Response

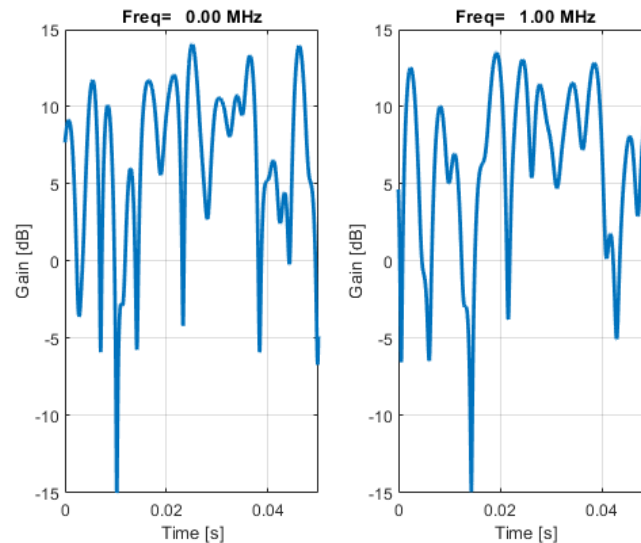
□ Narrowband response $H(t, f^0) = \sum_{\ell=1}^L g_{\ell} e^{j\omega_{\ell} t - j\omega^0 \tau_{\ell}}$

□ Plot at $f^0 = 0$ and 1 MHz

□ Max Doppler $f_{dmax} \approx 200$ Hz

- See fast variation on order $\frac{1}{f_{dmax}} = 5$ ms

□ Can see deep fades



```
% Compute the fading gain over time
t = linspace(0,0.05,1000)';
npath = length(gain);

% Random initial phase on each path at freq = 0
phi0 = 2*pi*rand(npath,1);

% Number of frequencies to test
ftest = [0,1e6];
nfreq = length(ftest);

for i=1:nfreq
    % Initial phase
    phi = phi0 -| 2*pi*ftest(i)*dly;

    % Gains on each path
    G = db2mag(gain).*exp(1i*(2*pi*fd*t' + phi) );

    % Sum over paths
    g = sum(G,1)';

    % Get power gain
    gpow = 20*log10(abs(g));

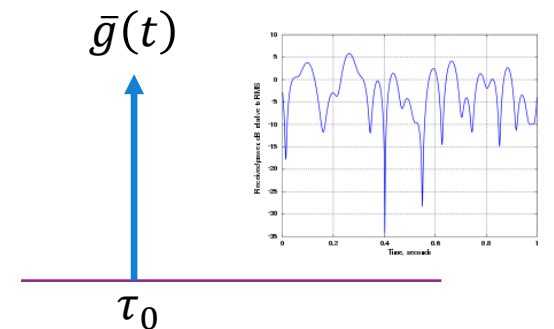
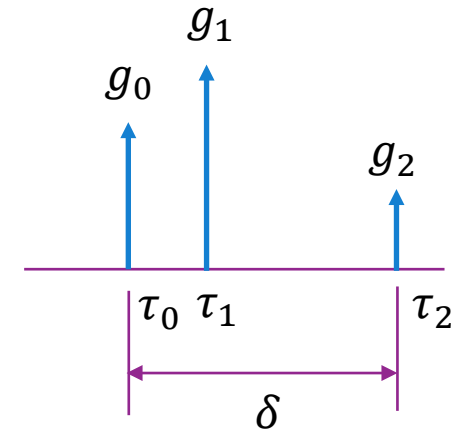
    subplot(1,nfreq,i);
    plot(t, gpow, 'Linewidth', 2);
    grid on;
    xlabel('Time [s]');
    ylabel('Gain [dB]');
    title(sprintf('Freq=%7.2f MHz', ftest(i)/1e6));
    ylim([-15,15]);
    xlim([0,0.05]);
end
```


Narrowband Approximation Proof

- Want to show: If $X(f)$ band-limited to $|f - f^0| \ll \frac{1}{2\delta}$ then $r(t) \approx H(t, f^0)x(t - \tau_0)$
- Prove this for $f^0 = 0$. Other frequencies are similar.
- Thus, $X(f)$ is bandlimited to $|f| \leq \frac{1}{2\delta}$.
- Therefore, $x(s_1) \approx x(s_2)$ for $|s_1 - s_2| \leq \delta$
- In particular $x(t - \tau_i) \approx x(t - \tau_0)$ since $|\tau - \tau_0| \leq \delta$
- Hence:

$$r(t) = \sum_{\ell=1}^L g_{\ell} e^{j\omega_{\ell} t} x(t - \tau_{\ell}) \approx \sum_{\ell=1}^L g_{\ell} e^{j\omega_{\ell} t} x(t - \tau_0)$$

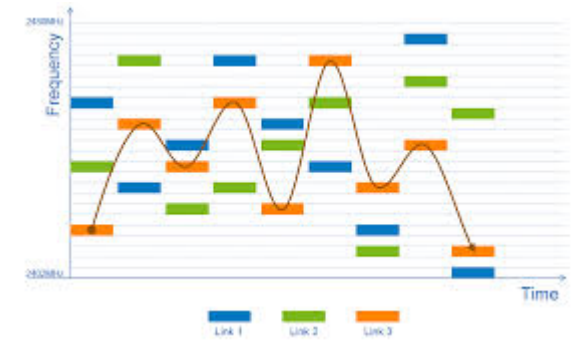
$$= H(t, 0)x(t - \tau_0)$$



Examples: When is Narrowband Valid?

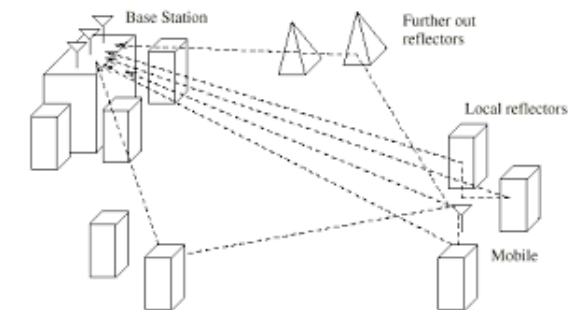
Bluetooth

- Bluetooth hops over channels of bandwidth $B = 1$ MHz each.
- Indoor delay spread typically $\ll 50$ ns
- Coherence bandwidth $\frac{1}{\delta} = \frac{1}{50} = 20$ MHz $\gg B$
- **Narrowband approximation valid**

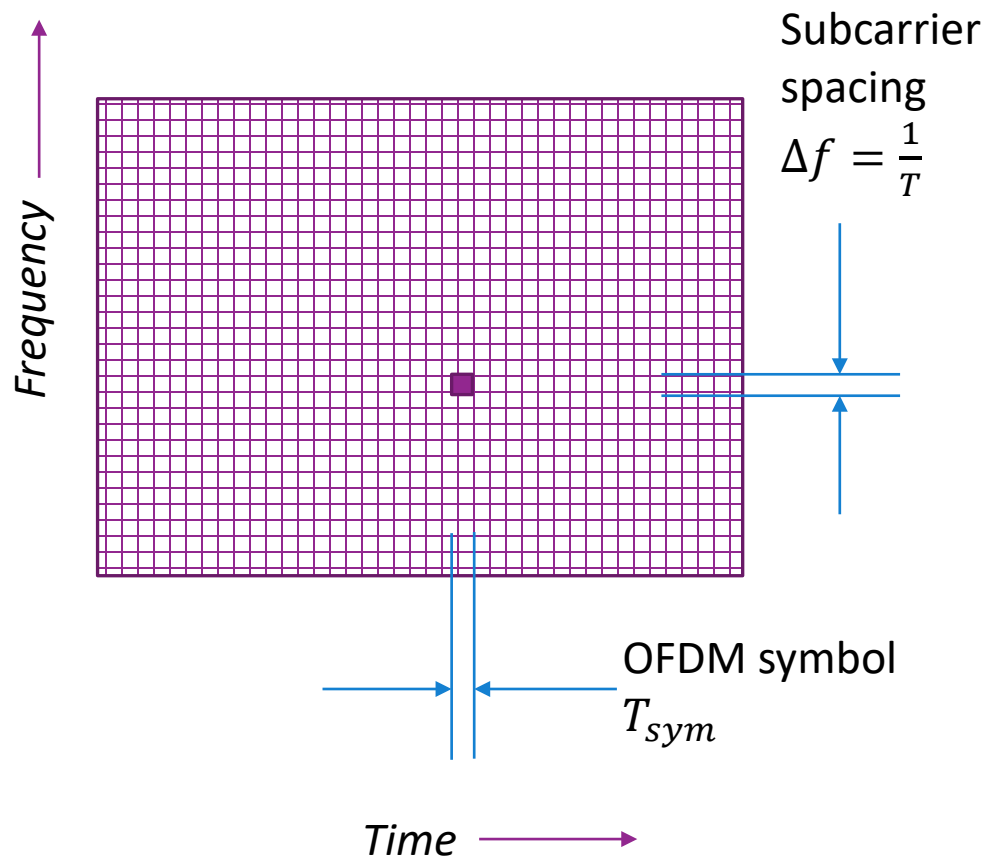


LTE outdoor cellular system

- A typical channel is $B = 20$ MHz
- Outdoor delay spread $\delta \approx 1$ μ s
- Coherence bandwidth $\frac{1}{\delta} = \frac{1}{1} = 1$ MHz $\ll B$
- **Narrowband approximation not valid**



OFDM Time-Frequency Grid



- ❑ OFDM modulation: Widely-used method
 - 4G and 5G cellular systems
 - Many 802.11 standards
- ❑ Divide channel into sub-carriers and OFDM symbols
 - **Resource element**: One time-frequency point
- ❑ Data is transmitted is an **array**: $X[n, k]$
 - k = OFDM symbol index
 - n = subcarrier index
 - One complex value per RE.
 - Called a **modulation symbol**
- ❑ See digital communication class
 - We will also review again when we discuss **equalization**

OFDM Channel with Fading

- OFDM channel acts as multiplication:
Under normal operation (delay spread is contained in CP):

$$Y[k, n] = H[k, n] X[k, n]$$

RX symbols Channel TX symbols

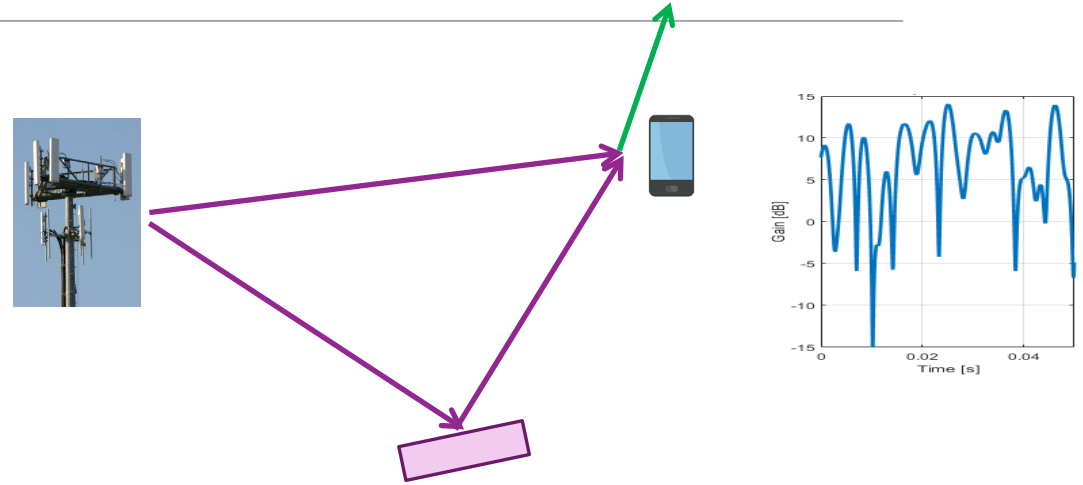
- OFDM channel gains can be computed from the multi-path components

$$H[k, n] = \sum_{\ell=1}^L \sqrt{E_{\ell}} e^{-2\pi j (Tkf_{\ell} + Sn\tau_{\ell} + \phi_{\ell})}$$

- T = OFDM symbol time, S = sub-carrier spacing
- For each path: f_{ℓ} = Doppler shift, τ_{ℓ} = Delay, ϕ_{ℓ} = phase of path, E_{ℓ} = path received energy

Summary

- ❑ Single path with no motion:
 - Delay and constant phase shift
- ❑ Local motion in single path causes Doppler
 - A time-varying phase rotation
 - But channel gain is constant
- ❑ Multiple paths causes fading
 - Constructive and destructive interference of paths
 - Variation in gain over time
- ❑ Described by a time-varying frequency response $H(t, f)$
 - Variations is time due to Doppler spread
 - Variations in frequency due to delay spread



In-Class Exercise: OFDM Channel Response

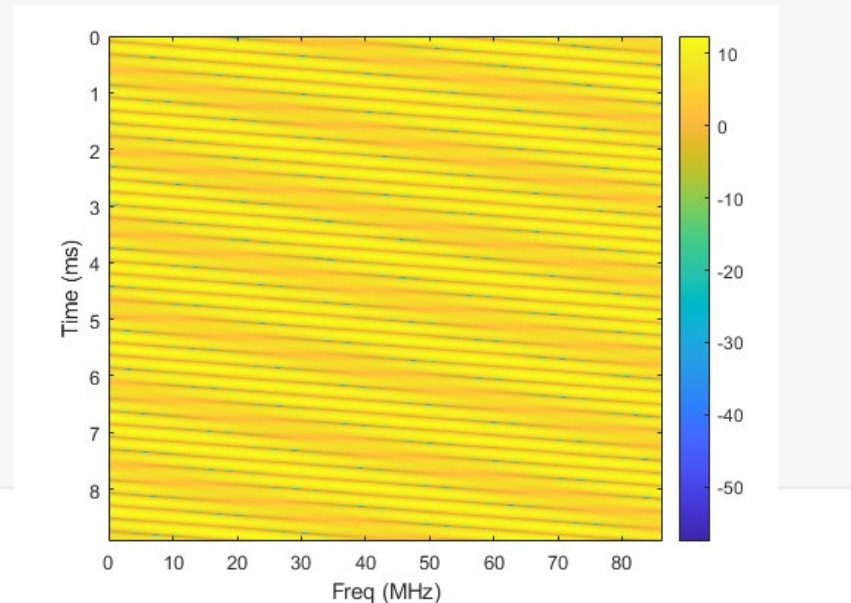
Problem 4: Computing an OFDM Frequency Response

Consider a system with the following parameters. These parameters are similar to common configuration for a 5G NR system used in the mmWave system

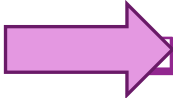
```
scs = 120e3; % sub-carrier spacing
nsc = 12*60; % number of sub-carriers
tsym = 1e-3/14/8; % OFDM symbol period
nsym = 1000; % number of symbols to plot

% Channel parameters
fc = 73e9; % carrier frequency
v = 10; % RX velocity in m/s
dly = [0,20,50]*1e-9; % Delay in sec of the paths
theta = [0,pi/4,pi]'; % Path AoA relative to motion
gaindB = [0,-3,-5]'; % gain of each path in dB

% Random initial phase of the gains
npath = length(dly);
phi = rand(npath,1)*2*pi;
```

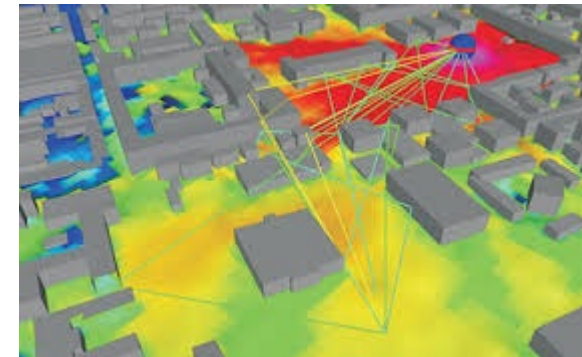
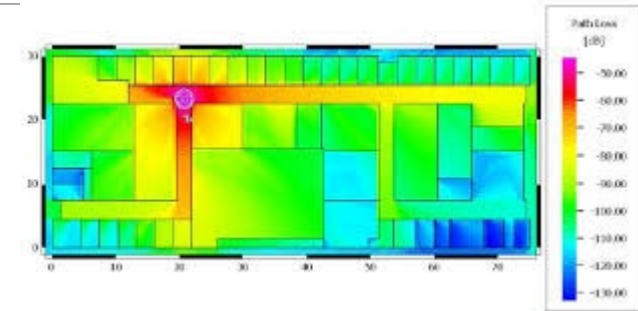


Outline

- Review of Up- and Downconversion
- Review of TX and RX Sampling
- Doppler and Multi-Path Fading
-  Statistical Descriptions of Fading

Statistical Model

- ❑ Fading depends on the multipath distribution
- ❑ Multipath is site-specific
- ❑ **Statistical model:**
 - Describes a **probability distribution** of channels
 - Trained on an ensemble of channels in some environment
- ❑ Used in evaluation of communication system
- ❑ Example questions:
 - How well does a system do *on average*?
 - What is the *probability* that I will obtain sufficient coverage?



Random Path Statistical Model

□ RX signal has many random, independent paths

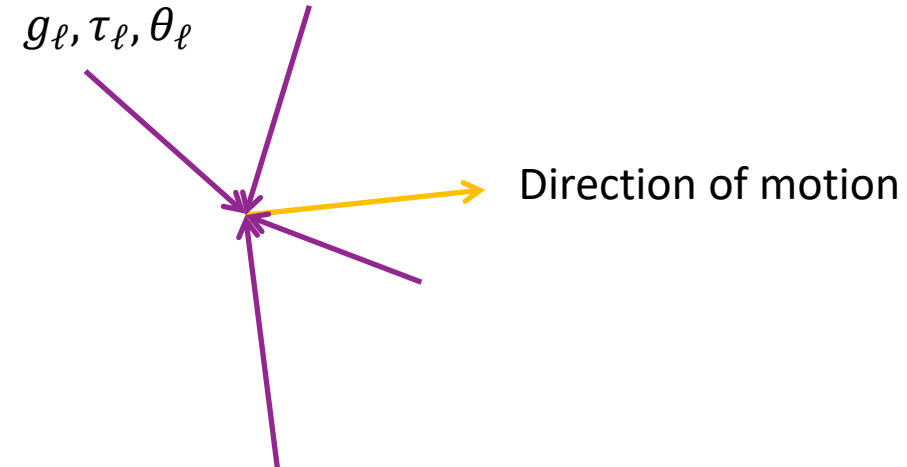
□ Time-varying frequency response:

$$h(t, f) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^L g_{\ell} e^{2\pi i(t f_d \cos \theta_{\ell} + f \tau_{\ell})}$$

- Assume $(g_{\ell}, \tau_{\ell}, \theta_{\ell})$ i.i.d.
- Path gains: g_{ℓ} are zero mean and $E|g_{\ell}|^2 = G_0$

□ As $L \rightarrow \infty$, $h(t, f)$ is a complex Gaussian, $h(t, f) \sim CN(0, G_0)$

- Follows from Central Limit Theorem
- Independent real and imaginary components
- G_0 : Average power gain
- Variance $G_0/2$ for real and imaginary components



Rayleigh Distribution

□ **Rayleigh fading**: Channel response is $h(t, f) \sim CN(0, G_0)$

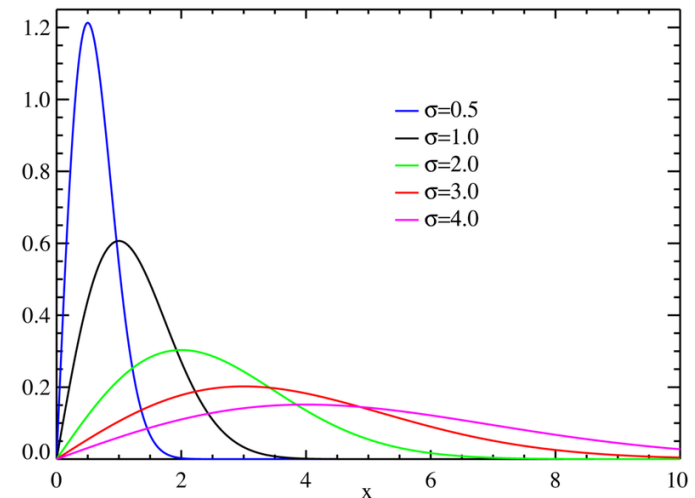
□ Let $R = |h|$ magnitude

- Represents amplitude gain

□ Has **Rayleigh distribution**:

- PDF: $p(r) = \frac{2r}{P} e^{-r^2/P}$
- CDF: $P(R \leq r) = 1 - e^{-r^2/P}$
- Mean: $E(R) = \sqrt{\frac{G_0\pi}{2}}$
- Second moment: $ER^2 = G_0$

Probability distribution



https://en.wikipedia.org/wiki/Rayleigh_distribution

Exponential Distribution

□ Consider Rayleigh fading complex gain $h \sim CN(0, G_0)$

□ Magnitude $R = |h|$ is Rayleigh

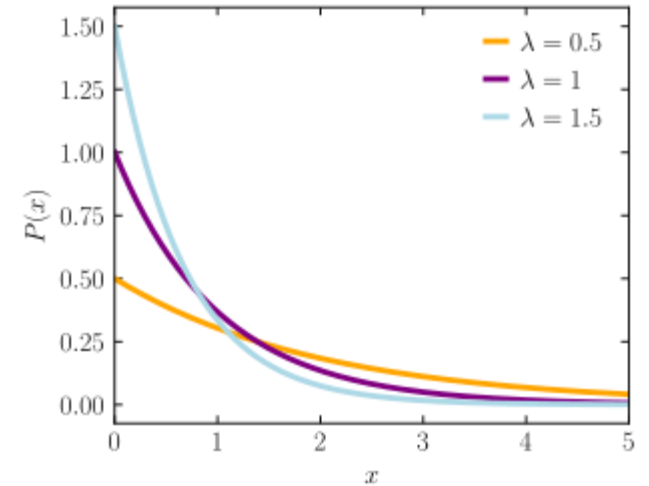
$$P(R \geq r) = e^{-r^2/G_0}$$

□ Instantaneous gain $G = |h|^2$ has exponential distribution

$$P(G \geq g) = P(R \geq \sqrt{g}) = e^{-g/G_0}$$

◦ Average gain is $E(G) = E|h|^2 = G_0$

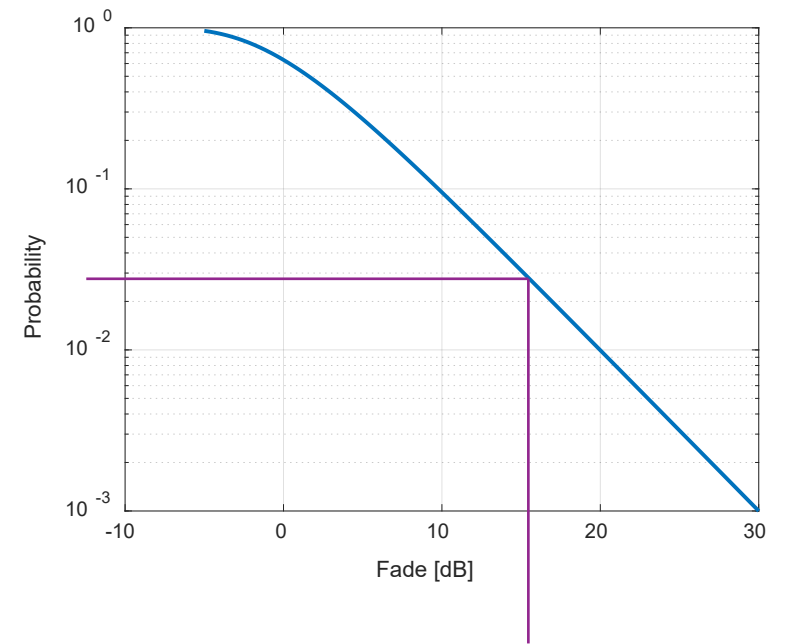
□ For channel, G represent power gain (in linear scale)



PDF for $\lambda = \frac{1}{E(G)}$

Example Calculation

- Suppose the channel experiences Rayleigh fading.
- What is probability gain will be 15 dB below the average?
 - Called a 15 dB fade.
- Answer:
 - Gain is 15 dB below average when $G \leq 10^{-0.1(15)} G_0$
 - From exponential distribution:
$$P(G \leq \beta G_0) = 1 - e^{-\beta G_0/G_0} = 1 - e^{-\beta}$$
 - For small β , $P(G \leq \beta G_0) \approx \beta$
 - For 15 dB fade, $\beta = 10^{-0.1(15)} \approx 0.032$.



Doppler Spectra

□ Consider statistical model:

$$h(t, f) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^L g_{\ell} e^{2\pi i(t f_d \cos \theta_{\ell} + f \tau_{\ell})}$$

- Paths are i.i.d. and g_{ℓ} are zero mean, $E|g_{\ell}|^2 = G_0$
- Assume L is large

□ For a given (t, f) , complex gain $h(t, f) \sim CN(0, G_0)$

□ As varies (t, f) , $h(t, f)$ is a **Gaussian random process**

□ Auto-correlation:

$$\begin{aligned} R(\delta t, \delta f) &= E[h(t, f)h^*(t + \delta t, f + \delta f)] \\ &= G_0 E\{e^{2\pi i(\delta t f_d \cos \theta_{\ell} + \delta f \tau_{\ell})}\} \end{aligned}$$

- Describes how correlated the process is over time and frequency
- Depends on the distribution of angles θ_{ℓ} and delays τ_{ℓ}

Jakes Model

At a fixed frequency f_0 :

$$h(t, f_0) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^L g_{\ell} e^{2\pi i (t f_d \cos \theta_{\ell} + f_0 \tau_{\ell})}$$

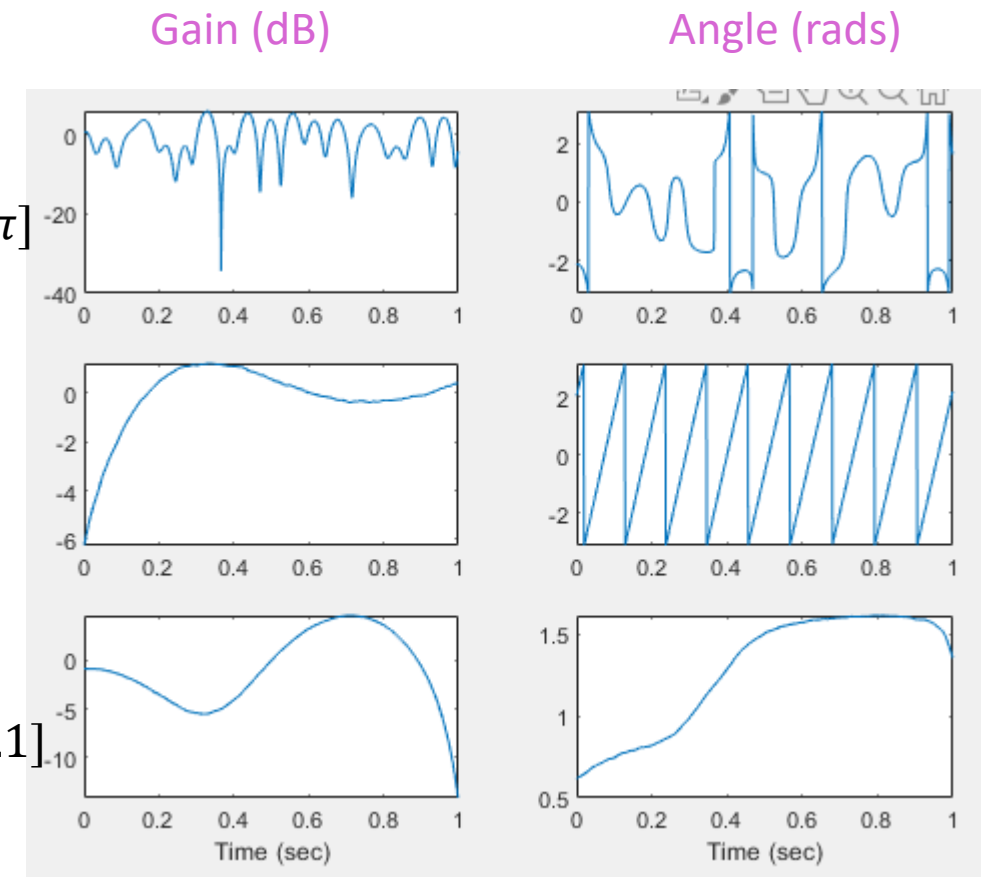
$$= \frac{1}{\sqrt{L}} \sum_{\ell=1}^L g'_{\ell} e^{2\pi i t f_d \cos \theta_{\ell}}$$

- Complex Gaussian process
- Statistics depend on angular distribution
- Jakes model:
 - Angles uniform from $[0, 2\pi]$
- Asymmetric Jakes:
 - $\theta \in [\theta_1, \theta_2]$ uniform
- Angular spread:
 - Arises from diffuse reflection

Jakes
Angles unif $[0, 2\pi]$

Asym Jakes
 $\cos \theta \in [0.9, 1]$

Asym Jakes
 $\cos \theta \in [-0.1, 0.1]$



Fading Models in MATLAB

- ❑ Comm Toolbox:
 - Efficient, general fading models
- ❑ Create a `comm.RayleighChannel` object
- ❑ Run the channel to get:
 - Output and gain

```
% Create Doppler models
nmod = 3;
dopMod = cell(nmod,1);
dopMod{1} = doppler('Jakes');
dopMod{2} = doppler('Asymmetric Jakes', [0.9 1]);
dopMod{3} = doppler('Asymmetric Jakes', [-0.1 0.1]);

% Simulate the channel gains for each model
]for i = (1:nmod)
    chan = comm.RayleighChannel(...
        'SampleRate', fsym, 'AveragePathGains', 0, ...
        'MaximumDopplerShift', fdmax, ...
        'DopplerSpectrum', dopMod{i}, ...
        'PathGainsOutputPort', true);

    [y, gain] = chan.step(x);
endfor
```

Auto-Correlation

- Fix a frequency f_0 and consider auto-correlation over time:

$$R(\delta t) = E[h(t, f_0)h^*(t + \delta t, f_0)] = G_0 E\{e^{2\pi i(\delta t f_d \cos \theta_\ell)}\}$$

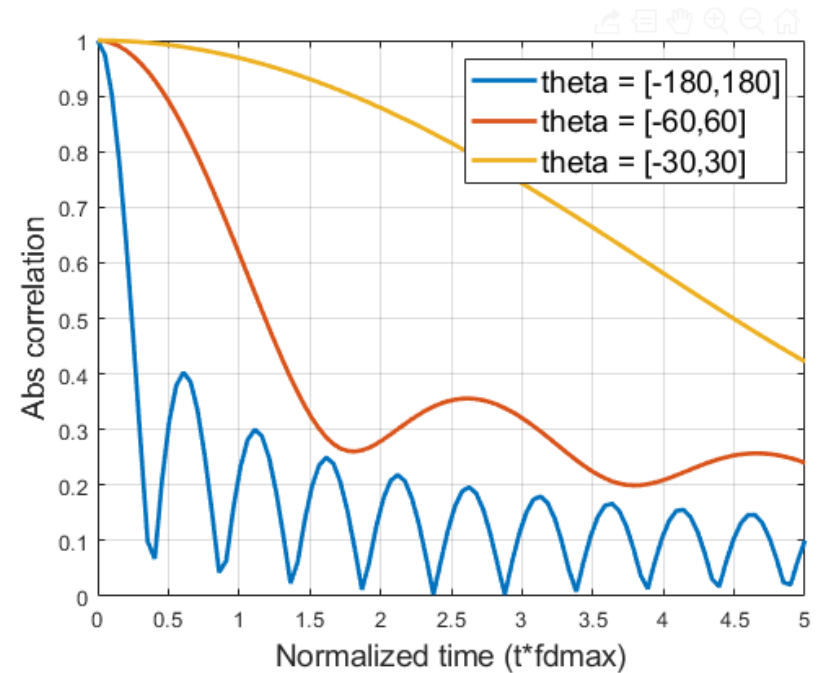
- Expectation is over angle θ_ℓ
- Depends on distribution of θ_ℓ

- Plot: $R(\delta t)$ for θ uniform in $[-\theta_m, \theta_m]$

- Computed numerically in MATLAB (see demo)
- Plotted vs normalized delay $f_d \delta t$
- For $\theta_m = 180^\circ \Rightarrow$ Jakes spectra
- Uncorrelated at $\delta t \approx \frac{1}{2f_d}$

- As angular distribution is smaller:

- Correlation is higher with delay
- Highly directional channel vary slower



Computing Auto-Correlation Numerically

Correlation in previous slide

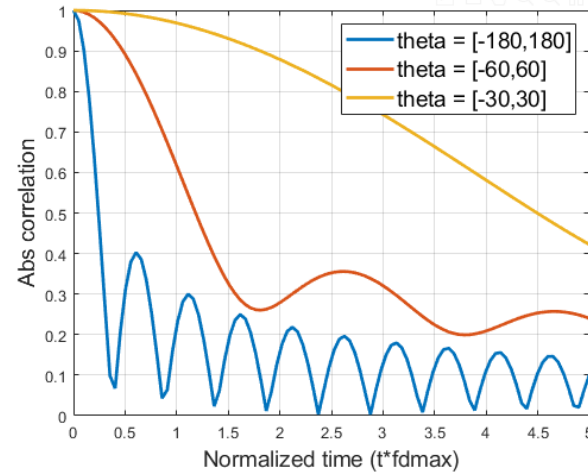
$$R(\delta t) = G_0 E\{e^{2\pi i(\delta t f_d \cos \theta_\ell)}\}$$

- Expectation is over angle θ_ℓ

Generally, no analytic solution

Compute via numerical integration

- See demo



```
% Function handle for the integrand
rfun = @(theta,tf) exp(1i*2*pi*cos(deg2rad(theta))*tf);

% Angle ranges to test
thetaMaxTest = [180, 60, 30];
ntheta = length(thetaMaxTest);

% Times to test:
t = linspace(0,5,100);
nt = length(t);

% Compute correlation for each angle range and time
Rcorr = zeros(nt, ntheta);
legStr = cell(ntheta,1);
for j = 1:ntheta
    thetam = thetaMaxTest(j);
    legStr{j} = sprintf('theta = [%d,%d]', thetam, thetam);
    for i = 1:nt
        Z1 = integral(@(theta) rfun(theta,t(i)), -thetam, thetam);
        Rcorr(i,j) = abs(Z1/(2*thetam));
    end
end

plot(t, Rcorr, 'LineWidth', 2);
grid on;
xlabel('Normalized time (t*fdmax)', 'FontSize', 14);
ylabel('Abs correlation', 'FontSize', 14);
legend(legStr, 'FontSize', 14);
```

Coherence Time and Frequency

□ Consider time varying frequency response $H(t, f)$

□ Coherence time:

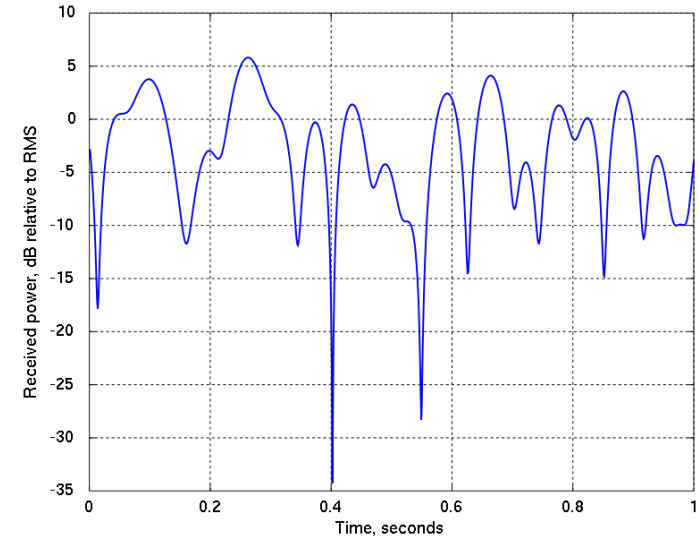
- Max interval Δt where $H(t, f) \approx H(t + \Delta t, f)$
- How fast channel changes in time
- Related to Doppler spread $\approx \frac{1}{f_{max} - f_{min}}$

□ Coherence bandwidth

- Max interval Δf where $H(t, f) \approx H(t, f + \Delta f)$
- How fast channel changes in frequency
- Related to delay spread $\approx \frac{1}{\tau_{max} - \tau_{min}}$

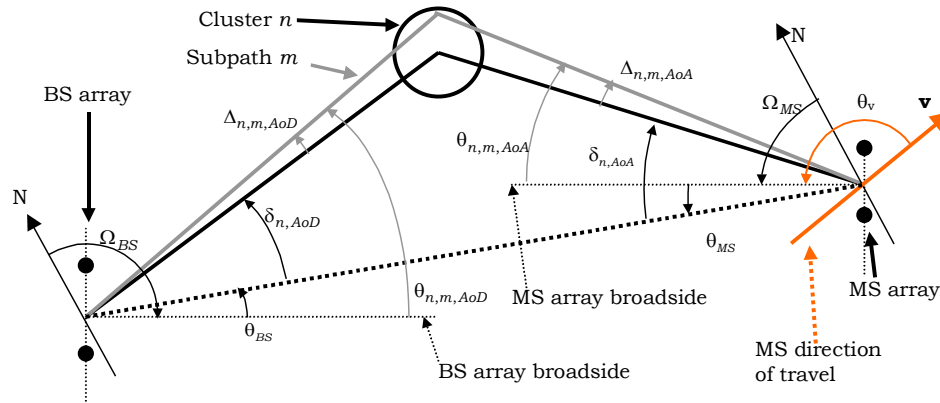
□ Critical for many procedures:

- Channel estimation, tracking, coding, ARQ, ...
- More on this later



Realization of a Jakes process with $1/f_{max} = 0.1$ sec

Winner-3GPP-Spatial Cluster Model

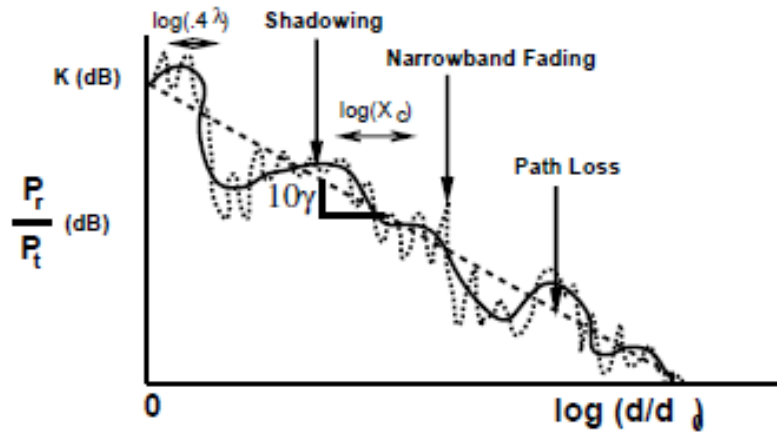


From 3GPP SCM-132

- ❑ Paths arrive in clusters.
- ❑ Clusters have subpaths (also called rays)
- ❑ Each cluster has:
 - Center angle and a statistical model for the delay and angular spread

Fading at Different Time Scales

Three mechanisms for path loss variations



Distance-based path loss

Shadowing

Small-scale multi-path fading

Slower

Faster

Figure 3.8: Combined Path Loss, Shadowing, and Narrowband Fading.

From Goldsmith, "Wireless Communications"

Fading at Different Scales Models

Source of variation	Mathematical model	Typical spatial coherence	Typical temporal coherence
Small-scale fading from multi-path fading	Rayleigh or Rician distribution	~ 1 wavelength	15 ms ($v=10\text{m/s}$, $f_c=2\text{GHz}$)
Large-scale fading from variations in shadowing	Lognormal distribution	10 to 100 m	1 to 10 sec
Path loss variations	Path loss exponent	100 m or larger	10 sec

- ❑ Different fading processes and variations occur at much different time / space scales
- ❑ Methods to combat these are different

