Problems: Antenna Arrays and Beamforming
ECE-GY 6023. Wireless Communications

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1. *ULA.* Consider a ULA with spatial signature,

$$\mathbf{u}(\phi) = \left[1, e^{i\beta\sin(\phi)}, \cdots, e^{i\beta(N-1)\sin(\phi)}\right]^{\intercal}, \quad \beta = \frac{2\pi d}{\lambda},$$

where ϕ is the azimuth angle relative to boresight.

   (a) Find the optimal beamforming vector for an angle $\phi = 30^\circ$ with $N = 8$ antennas and $d = \lambda/2$.

   (b) Now suppose a path arrives at an angle $\phi = 60^\circ$. What is the beamforming gain on that path using the BF vector in part (a)?

2. *Constrained beamforming.* Consider a MISO channel,

$$\mathbf{r} = \mathbf{h}x + \mathbf{v}, \quad \mathbf{v} \sim \mathcal{CN}(0, N_0\mathbf{I}), \quad |x|^2 = E_x.$$

   (a) What is the SNR after beamforming with a vector $\mathbf{w}$?

   (b) What is the maximum SNR after beamforming if we are allowed any vector $\mathbf{w}$?

   (c) What is the maximum SNR after beamforming if the components of $\mathbf{w}$ must be constant magnitude, $|w_n| = 1$. That is, you can only change the phase of w_n. This commonly occurs in analog beamforming systems.

   (d) What is the maximum SNR after beamforming if $w_n \neq 0$ for only one antenna n. This is called antenna selection beamforming.

   (e) Suppose $\mathbf{h} = [4, 2+i, -1, i]^{\intercal}$ and $E_x/N_0 =$ 5dB. What is the SNR after beamforming in parts (b), (c) and (d)?

3. *Vector frequency response.* Consider a MISO channel where

$$\mathbf{r} = \mathbf{h}(f)\mathbf{x} + \mathbf{v}, \quad \mathbf{v} \sim \mathcal{CN}(0, N_0\mathbf{I}), \quad |x|^2 = E_x,$$

where $\mathbf{h}(f)$ is the frequency-dependent channel

$$\mathbf{h} = \sum_{\ell=1}^{L} a_\ell e^{2\pi i f \tau_\ell}\mathbf{u}(\theta_\ell),$$

and L is the number of paths. Suppose the array is $N = 8$ ULA with antenna spacing $d = \lambda/2$ and the channel has three paths with parameters shown in Table 1. In the table, the column "SNR" is $E_x|a_\ell|^2/N_0$, and the column "path phase" is the angle of a_ℓ.

Write a short MATLAB program to plot the following SNR values for 100 frequency points in the range $f \in [-10, 10]$ MHz:

Path	SNR [dB]	Path phase [deg]	AoA ϕ_ℓ [deg]	Delay τ_ℓ [ns]
1	4	0	30	0
2	1	180	-30	100
3	-2	65	80	130

Table 1: Problem 3: Path parameters.

(a) The SNR only using the signal from antenna 1.

(b) The SNR after beamforming where the beamforming vector is optimally selected at each frequency.

(c) The SNR after beamforming where the beamforming vector is optimized for $f = 0$.

4. *Beamforming with SVDs.* Below are a set of channel matrices $\mathbf{H}$ described by TX and RX spatial signatures $\mathbf{u}_{\rm tx}(\Omega_\ell^{\rm tx})$ and $\mathbf{u}_{\rm rx}(\Omega_\ell^{\rm rx})$ and complex gains g_ℓ. For each of the channel matrices, find the rank r, the maximum singular value and the optimal TX and RX beamforming vectors. Assume all matrices have dimensions $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$, and the spatial signatures are normalized so that

$$\|\mathbf{u}_{\rm tx}(\Omega^{\rm tx})\|^2 = N_t, \quad \|\mathbf{u}_{\rm rx}(\Omega^{\rm rx})\|^2 = N_r,$$

for all angles $\Omega^{\rm tx}$ and $\Omega^{\rm rx}$.

(a) Single path channel:

$$\mathbf{H} = g_1 \mathbf{u}_{\rm rx}(\Omega_1^{\rm rx})\mathbf{u}_{\rm tx}(\Omega_1^{\rm tx}).$$

(b) Two channel paths with same RX angle $\Omega_1^{\rm rx}$ and two different TX angles, $\Omega_1^{\rm tx}$ and $\Omega_2^{\rm tx}$:

$$\mathbf{H} = g_1 \mathbf{u}_{\rm rx}(\Omega_1^{\rm rx})\mathbf{u}_{\rm tx}(\Omega_1^{\rm tx}) + g_2 \mathbf{u}_{\rm rx}(\Omega_1^{\rm rx})\mathbf{u}_{\rm tx}(\Omega_2^{\rm tx}).$$

Assume $|g_1| > |g_2|$ and $\mathbf{u}_{\rm tx}(\Omega_1^{\rm tx}) \perp \mathbf{u}_{\rm tx}(\Omega_2^{\rm tx})$

(c) Two channel paths with two different RX angles and two different TX angles:

$$\mathbf{H} = g_1 \mathbf{u}_{\rm rx}(\Omega_1^{\rm rx})\mathbf{u}_{\rm tx}(\Omega_1^{\rm tx}) + g_2 \mathbf{u}_{\rm rx}(\Omega_2^{\rm rx})\mathbf{u}_{\rm tx}(\Omega_2^{\rm tx}).$$

Assume $|g_1| > |g_2|$, $\mathbf{u}_{\rm rx}(\Omega_1^{\rm rx}) \perp \mathbf{u}_{\rm rx}(\Omega_2^{\rm rx})$ and $\mathbf{u}_{\rm tx}(\Omega_1^{\rm tx}) \perp \mathbf{u}_{\rm tx}(\Omega_2^{\rm tx})$.

(d) Same as part (b) except the two TX spatial signatures are not orthongonal:

$$\mathbf{u}_{\rm tx}(\Omega_1^{\rm tx})^* \mathbf{u}_{\rm tx}(\Omega_2^{\rm tx}) = \rho N_t,$$

for some ρ with $|\rho| \leq 1$.

5. *Array normalization.* When there is no mutual coupling, an array with N elements can generally obtain a gain of N (in linear scale). However, in this exercise, we will show that the gain is more limited when the antennas are closely spaced. To this end, consider a ULA with N elements with total length L, so the antennas are spaced by L/N. Given a beamforming vector $\mathbf{w} \in \mathbb{C}^N$, suppose the power intensity at angle ϕ is,

$$U(\phi) = c|\mathbf{w}^\mathsf{T}\mathbf{u}(\phi)|^2,$$

where $\mathbf{u}(\phi)$ is the spatial signature and $c > 0$ is some constant. Suppose we use the all ones beamforming vector $w_n = \frac{1}{N}$ so that the energy is maximized at $\phi = 0$.

(a) Write the spatial signature $\mathbf{u}(\phi)$ as a function of the total array length L, wavelength λ and number of elements N.

(b) Find the limit

$$\lim_{N \to \infty} U(\phi).$$

(c) Recall the antenna directivity is

$$D(\phi) = \frac{4\pi U(\phi)}{P_{\text{rad}}},$$

where P_{rad} is the total radiated power,

$$P_{\text{rad}} = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} U(\phi) \cos\theta \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

Use MATLAB to plot $D(\phi)$ using the limit of large numbers of antennas for $L = 2\lambda$, 4λ and 8λ. What is the peak gain in each case.