Multi-Path Fading

WIRELESS SHORT COURSE

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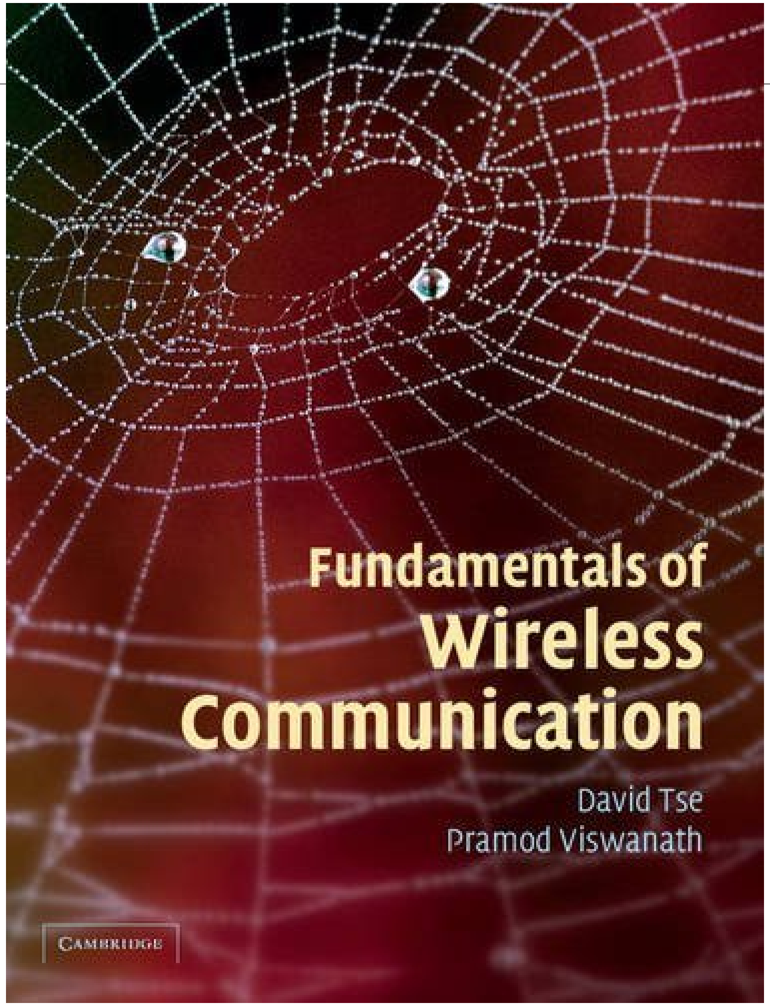
Fading

From the Introduction of a classic text:

There are two fundamental aspects of wireless communication that make the problem challenging and interesting.

…First is the phenomenon of fading …

…Second …there is significant interference …



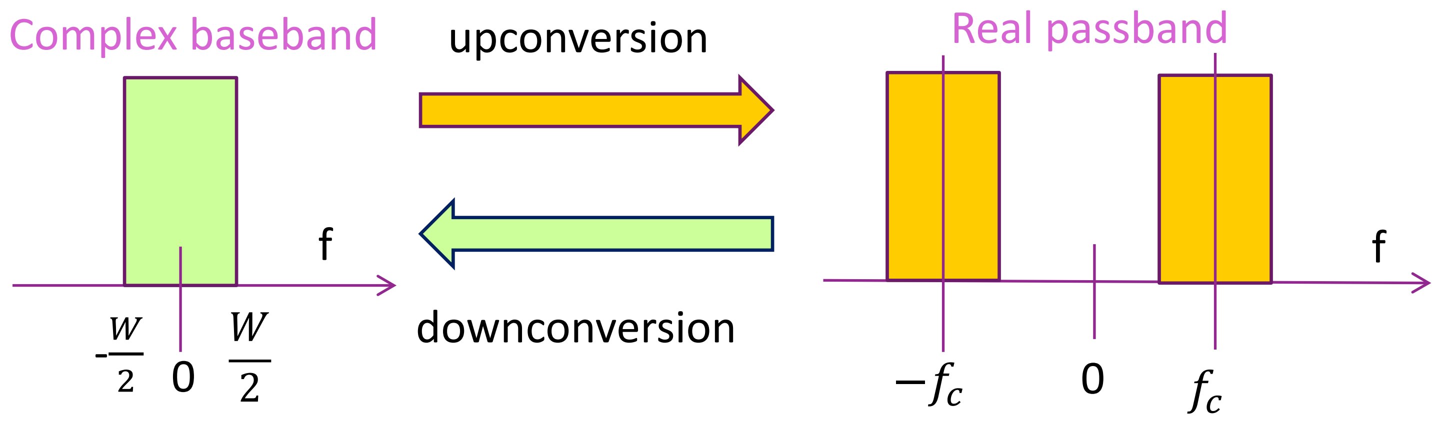
Learning Objectives

- Describe up and down-conversion in time- and frequency-domain
- Describe the steps in the DAC and ADC including the filtering
- Compute a discrete-time and continuous-time base equivalent channels from the passband
- Simulate fractional delays and gains in the sampled data
- Describe and simulate a deterministic multi-path wireless channel
- Compute the time-varying frequency response given the path parameters
- Describe a statistical model for multi-path fading
- Approximately compute the coherence time and bandwidth given a channel

Outline

- Review of Up- and Downconversion
- Review of TX and RX Sampling
- Doppler and Multi-Path Fading
- Statistical Descriptions of Fading

Up- and Downconversion



- RF communication systems:
  - Information occurs and is processed in complex baseband
  - Transmitted and received in real passband
- Up and down-conversion: Shift center frequency of signals
- Also called mixing

Up and Down-Conversion in Time Domain

- Complex baseband:
  - Two real signals, $u_I(t), u_Q(t)$
  - Or, one complex signal:

$$u(t) = u_I(t) + ju_Q(t)$$

upconversion

$$u_p(t) = Re(u(t)e^{j\omega_c t})$$

- Real passband: $u_p(t)$

downconversion

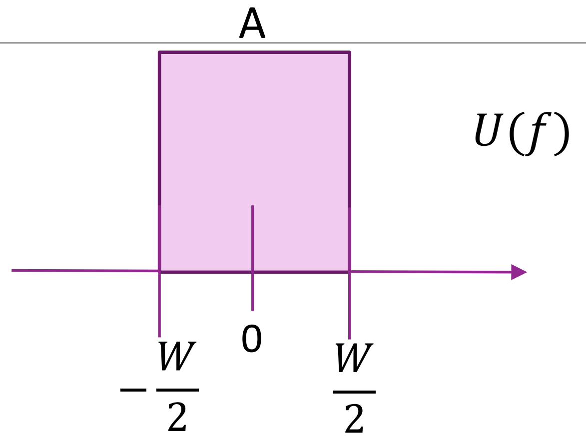
$$v(t) = 2u_p(t)e^{-j\omega_c t}$$
$$u(t) = h_{LPF}(t) * v(t)$$

Note: downconversion needs:

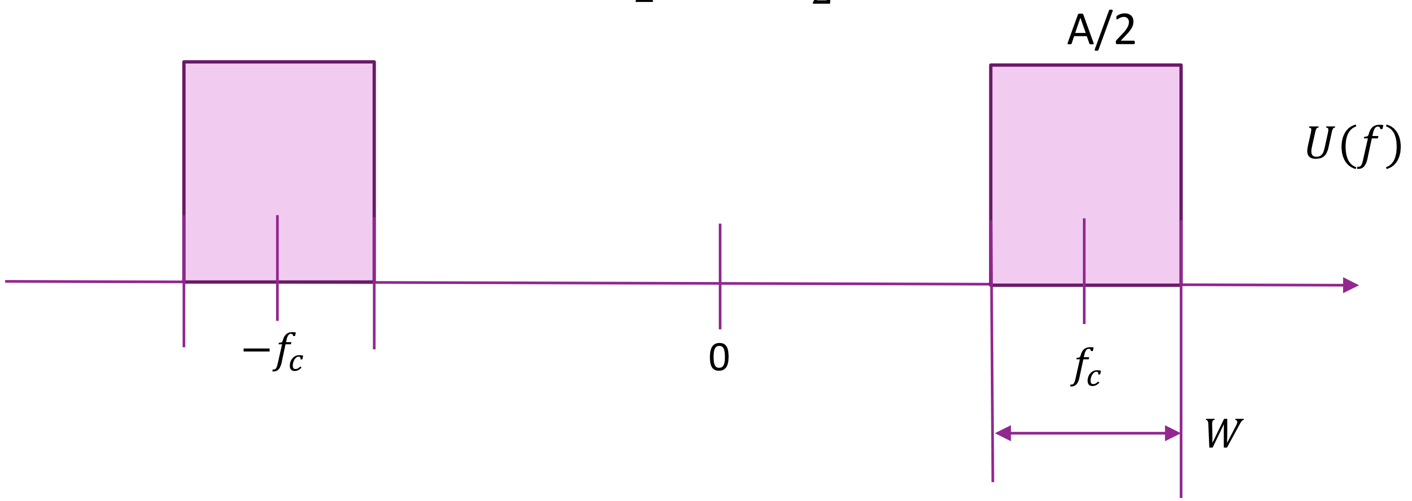
- Multiplication by 2
- Low pass filtering

Mixing in Frequency Domain

- Baseband signals
  - Centered around $f = 0$, complex
  - $\frac{W}{2}$ = single sided bandwidth
  - W = two sided bandwidth
  - Band-limited to $|f| \le \frac{W}{2}$



- Passband signals
  - Centered around $f = f_c$, real
  - W = bandwidth (per side or image)
  - Band-limited to $|f - f_c| \le \frac{W}{2}$



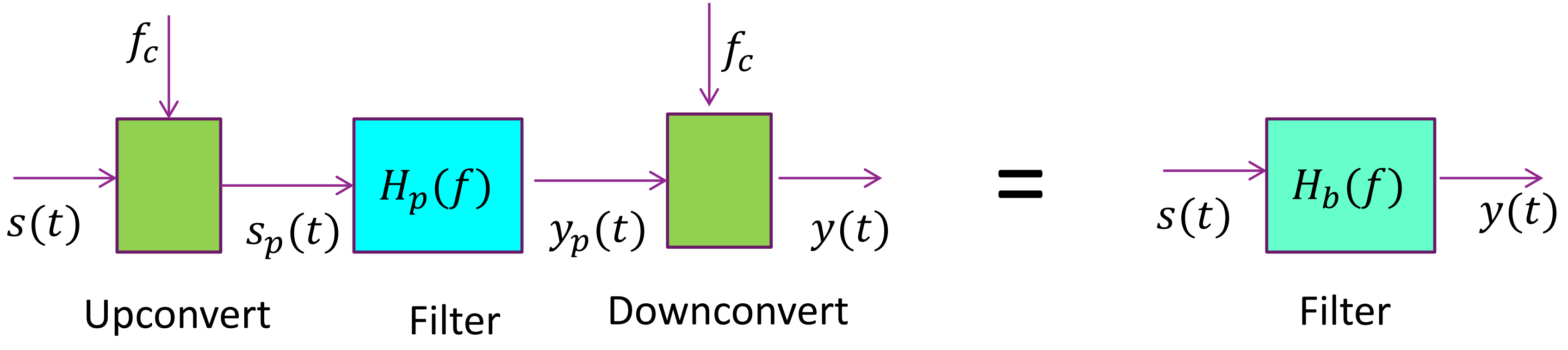
Discrete IQ Mixer

- LO = “local oscillator” = square or sine wave at f_c
- I1, I2 = I and Q inputs.
  - Generally, lowpass
- RF = passband output centered at f_c

http://www.markimicrowave.com/Mixers/IQ_Quadrature-IF_Double-Balanced/IQ-0318.aspx

Datasheet	RF [GHz]	LO [GHz]	IF [MHz]	Conversion Loss [dB]	Image Rejection [dB]	Amplitude Deviation [dB]	Phase Deviation [Degrees]	Isolations L-R [dB]	Isolations L-I [dB]
IQ-0318	3 to 18	3 to 18	DC to 500	7	22	0.75	10	40	20

Baseband Equivalent Channel



- Filtering at passband equivalent to complex baseband filter
- Assuming downconversion filter is ideal:
  - $H_b(f) = H_p(f + f_c)$ for $|f| \leq \frac{W}{2}$
  - Simply shift $H_p(f)$ to the left by f_c.

Important Special Case: Delay

Passband channel

$$h_p(t) = A\delta(t-\tau)$$

A = gain
τ = delay

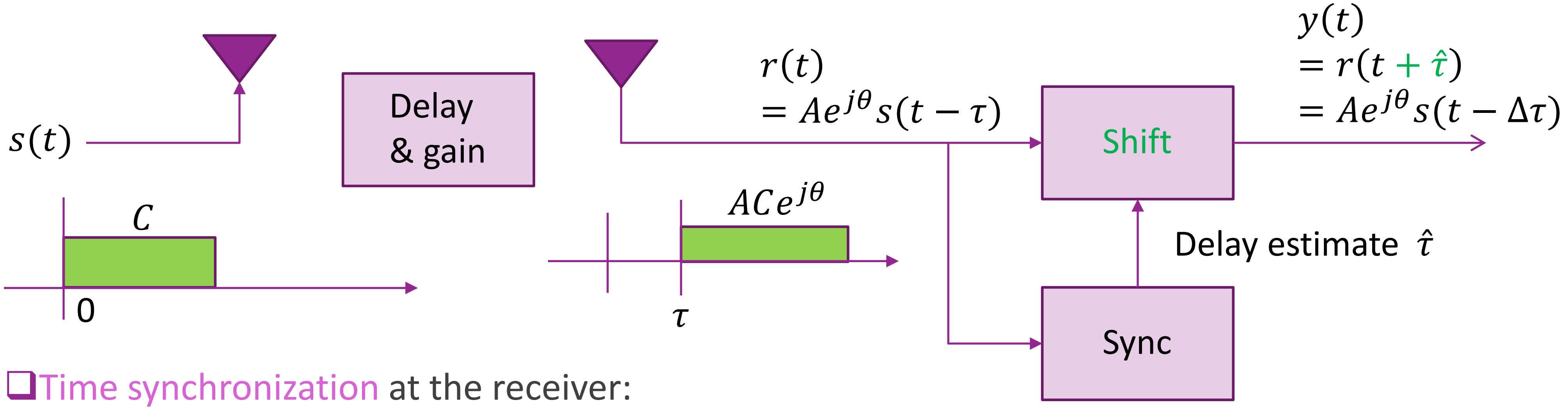
Baseband equivalent channel

$$h_b(t) = Ae^{-2\pi j f_c \tau}\delta(t-\tau)$$

A = gain
τ = delay
$\theta = 2\pi f_c \tau$ = phase rotation

- Delay, gain in passband ⇒ delay, gain and phase rotation in baseband
- Proof:
  - Passband frequency response is: $H_p(f) = Ae^{-2\pi j f \tau}$
  - Baseband frequency response: $H_b(f) = H_p(f+f_c) = Ae^{-2\pi j (f_c + f)\tau}$
  - Equivalent impulse response: $h_b(t) = Ae^{-2\pi j f_c \tau}\delta(t-\tau)$

Synchronization and Delay Errors



- Time synchronization at the receiver:
  - Estimate the arrival time of the signal $\hat{\tau}$
  - Starts processing remainder of signal starting at $\hat{\tau}$
  - Equivalent to shifting received signal ahead in time by $\hat{\tau}$: $y(t) = r(t + \hat{\tau})$
  - Remaining time error: $\Delta\tau = \tau - \hat{\tau}$
- Later, we will discuss:
  - How to estimate τ (synchronization) and how to correct for gain and phase error (equalization)

In-Class Problem

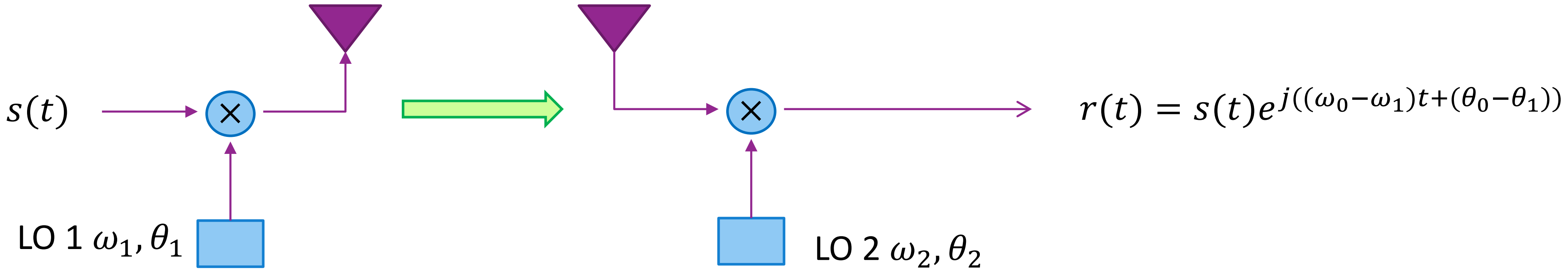
Up and down-conversion, Problem 1

Consider a system with the following parameters

```
bw = 20e6;     % bandwdith
pl = 80;       % path loss (dB)
tau = 200e-9;     % timing error

% TODO:  Plot the real component of the frequency response
% over the bandwidth.  Assume phase = 0 at DC.
% Use 1024 frequency points
```

Frequency Errors



- Oscillators at TX and RX always have some mismatch. To analyze, suppose:
  - Upconversion: $s_p(t) = Re\left(s(t)e^{j\omega_1 t+\theta_1}\right)$,
  - Downcoversion: $r(t) = LPF(\, 2s_p(t)e^{-(j\omega_2 t+\theta_2)}\,)$
- LO error leads to time-varying gain: $r(t) = g(t)s(t), \;\; g(t) = e^{j((\omega_0-\omega_1)t+(\theta_0-\theta_1))}$
  - Frequency and phase shift

In-Class Problem

Up and down-conversion, Problem 2

Suppose a link has the following parameters:

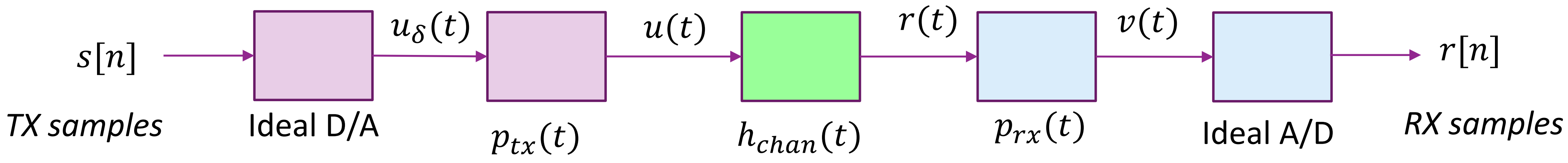
```
fc = 37e9;   % carrier freq
loppm = 1;   % LO error in ppm

% TODO:  Plot the relative change of the gain
%
%     E(t) = |g(t)-g(t+tau)|^2 / |g(t)|^2
%
% as a function of tau from 0 to 5 us
```

Outline

- ❑Review of Up- and Downconversion
- ❑Review of TX and RX Sampling
- ❑Doppler and Multi-Path Fading
- ❑Statistical Descriptions of Fading

Typical Digital Communication Path



- All modern communication systems TX and RX digital samples
- Transmitter: DAC + filtering with $p_{tx}(t)$. Filtering used to:
  - Suppress out of band emissions
- Receiver: Filters with $p_{rx}(t)$ then performs ADC. Filtering plays two roles:
  - Reduces noise
  - Remove out-of-band signals before ADC. (i.e. Anti-aliasing)
- Filter design discussed in digital communications class

Review of DTFT

- Given discrete-time sequence $s[n]$
  - Real or complex
- Discrete-time Fourier Transform: $S(\Omega) = \sum_n s[n]e^{-j\Omega n}$
- Inverse DTFT: $s[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\Omega)e^{j\Omega n}d\Omega$
- Note $S(\Omega)$ is always a 2π periodic signal
  - Can take integral for inverse DTFT on any period of 2π
- Ω is the discrete frequency. Units is radians per sample.
- For finite length signals and finite number of Ω, can be computed via FFT
- Review in your signals and systems class

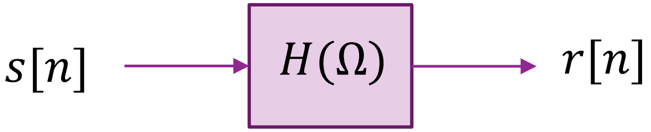
Common DTFT Pairs

Time domain $x[n]$	Frequency domain $X_{2\pi}(\omega)$
$\delta[n]$	$X_{2\pi}(\omega) = 1$
$\delta[n-M]$	$X_{2\pi}(\omega) = e^{-i\omega M}$
$\sum_{m=-\infty}^{\infty} \delta[n-Mm]$	$X_{2\pi}(\omega) = \sum_{m=-\infty}^{\infty} e^{-i\omega Mm} = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{M}\right)$ $X_o(\omega) = \frac{2\pi}{M} \sum_{k=-(M-1)/2}^{(M-1)/2} \delta\left(\omega - \frac{2\pi k}{M}\right)$ odd M $X_o(\omega) = \frac{2\pi}{M} \sum_{k=-M/2+1}^{M/2} \delta\left(\omega - \frac{2\pi k}{M}\right)$ even M
$u[n]$	$X_{2\pi}(\omega) = \frac{1}{1-e^{-i\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ $X_o(\omega) = \frac{1}{1-e^{-i\omega}} + \pi \cdot \delta(\omega)$
$a^n u[n]$	$X_{2\pi}(\omega) = \frac{1}{1-ae^{-i\omega}}$
e^{-ian}	$X_o(\omega) = 2\pi \cdot \delta(\omega + a)$, $-\pi < a < \pi$ $X_{2\pi}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega + a - 2\pi k)$

- See Wikipedia

$\cos(a \cdot n)$	$X_o(\omega) = \pi\left[\delta(\omega - a) + \delta(\omega + a)\right],$ $X_{2\pi}(\omega) \triangleq \sum_{k=-\infty}^{\infty} X_o(\omega - 2\pi k)$
$\sin(a \cdot n)$	$X_o(\omega) = \frac{\pi}{i}\left[\delta(\omega - a) - \delta(\omega + a)\right]$
$\operatorname{rect}\left[\frac{n - M/2}{M}\right]$	$X_o(\omega) = \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-\frac{i\omega M}{2}}$
$\operatorname{sinc}(W(n+a))$	$X_o(\omega) = \frac{1}{W} \operatorname{rect}\left(\frac{\omega}{2\pi W}\right) e^{ia\omega}$
$\operatorname{sinc}^2(Wn)$	$X_o(\omega) = \frac{1}{W} \operatorname{tri}\left(\frac{\omega}{2\pi W}\right)$

Discrete-Time Systems



- ❑Consider discrete-time LTI system
- ❑Time-domain: Characterized by impulse response $h[n]$

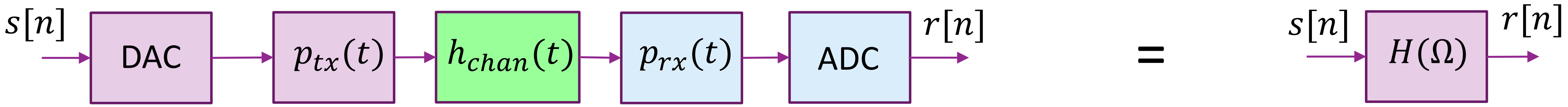
$$r[n] = h[n] * s[n] = \sum_k h[k]s[n-k]$$

- ❑Frequency-domain: Characterized by frequency response $H(\Omega)$

$$R(\Omega) = H(\Omega)S(\Omega)$$

  - $R(\Omega) = \sum r[n]e^{-j\Omega n},\ r[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R(\Omega)e^{j\Omega n}d\Omega$

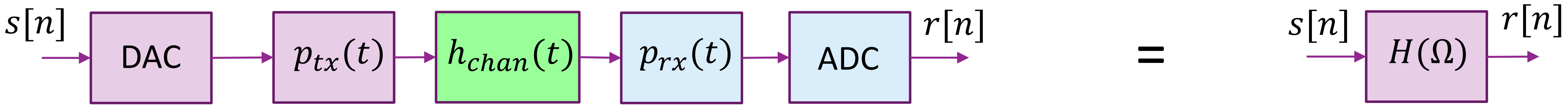
DT Equivalent Channel



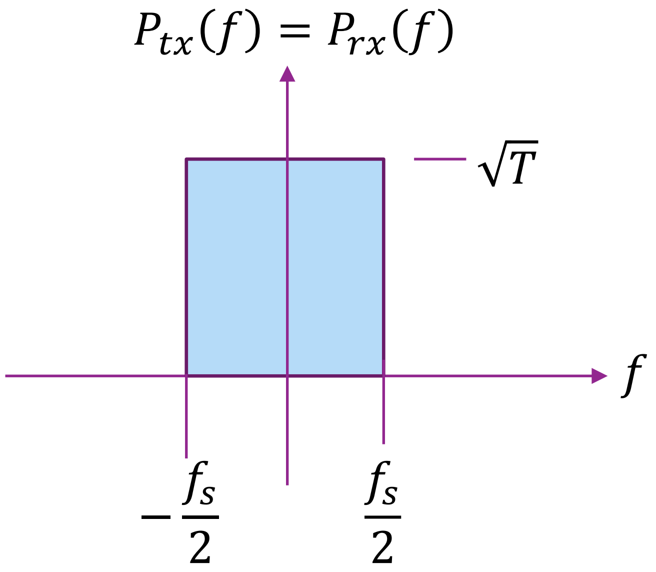
- **Discrete-time baseband equivalent channel**:
  - Describes equivalent mapping from $s[n]$ to $r[n]$
  - Includes effects of TX and RX filtering and continuous-time baseband channel
- **Band-limited filters**:
  - Suppose one of P_{rx}, P_{tx} is bandlimited to $|f| < \frac{1}{2T}$ (no out-of-band emissions or aliasing)
  - Then, discrete-time equivalent channel reduces to:

$$H(\Omega) = \frac{1}{T} P_{rx}\left(\frac{\Omega}{2\pi T}\right) P_{tx}\left(\frac{\Omega}{2\pi T}\right) H_{chan}\left(\frac{\Omega}{2\pi T}\right) \text{ for } |\Omega| < \pi$$

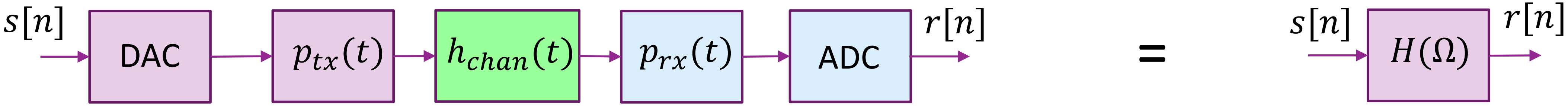
Ideal Filtering



- Suppose sample rate $f_s = \frac{1}{T}$
- "Ideal" TX and RX filter :
  - $p_{tx}(t) = p_{rx}(t) = \frac{1}{\sqrt{T}}\text{Sinc}\left(\frac{t}{T}\right)$
  - In frequency domain: $P_{rx}(f) = P_{tx}(f) = \sqrt{T}\text{Rect}(fT)$
  - Also called "brick wall" filter
- Most practical filters match this well
  - Up to gain and delay



Ideal Filtering



- Assume TX and RX filters are ideal
- Theorem: DT equivalent channel is the re-scaled continuous-time channel

$$H(\Omega) = H_{chan}\left(\frac{\Omega}{2\pi T}\right)$$

  - Frequency f mapped to $\Omega = 2\pi T f$

Special Case: Delay

	Passband	Continuous-Time Baseband	Discrete-Time Baseband
Impulse response	$h_p(t) = A\delta(t-\tau)$	$h_{chan}(t) = Ae^{-j\omega_c\tau}\delta(t-\tau)$	$h[n] = Ae^{-j\omega_c\tau} sinc\left(\frac{\tau n}{T}\right)$
Frequency response	$H_p(f) = Ae^{-j2\pi f\tau}$	$H_{chan}(f) = Ae^{-j\omega_c\tau}e^{-j2\pi f\tau}$	$H(\Omega) = Ae^{-j\omega_c\tau}e^{-j\Omega\tau/T}$

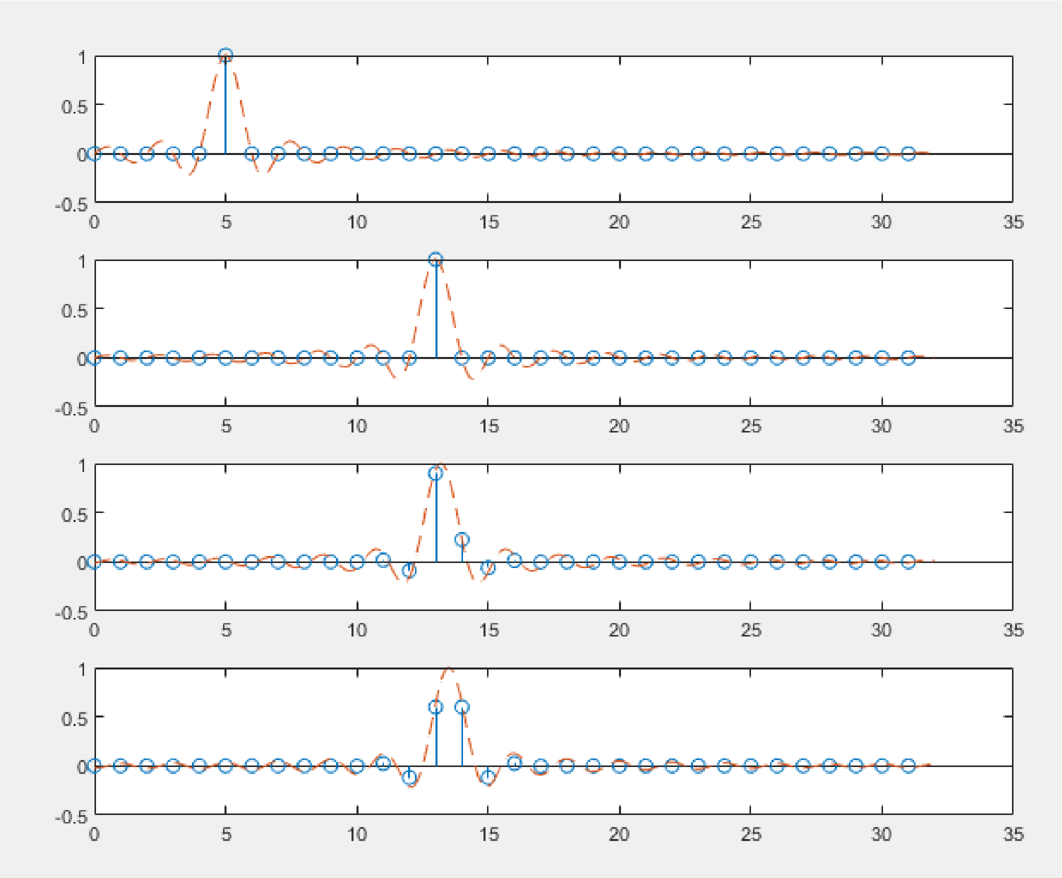
- Suppose passband has a gain and delay.
- Then discrete-time frequency-domain: gain and linear phase rotation over frequency
  - Rotates $2\pi\ \tau/T$ radians every period
- In discrete-time time-domain: gain, constant phase rotation and sinc filter with delay

Sinc Filter with Integer Delays

- Suppose we have ideal filtering and passband has delay and gain
- From previous slide, $r[n] = h[n] * s[n],\ h[n] = Ae^{-j\omega_c \tau} sinc\left(\frac{\tau n}{T}\right)$
- Special case 1: No delay $\tau = 0$:
  - $h[n] = A\delta[n] \Rightarrow\ r[n] = As[n]$
  - Baseband channel introduces only gain
- Special case 2: Integer delays $\tau = kT$:
  - $h[n] = A\delta[n-k] \Rightarrow\ r[n] = As[n-k]$
  - Baseband channel introduces gain and integer shift
- Ex: Suppose sample rate is 20 MHz and signal is delayed by 400 ns.
  - Integer delay in discrete-time signal is $\frac{20}{0.4} = 50$ samples

Sinc Pulses with Fractional Delay

- $h[n] = Ae^{-j\omega_c \tau} sinc\left(\frac{\tau n}{T}\right)$
- Causes blurring over multiple samples
- Inter-symbol interference
- Will need equalization to correct
  - More on this later



$\tau = 0$

$\tau = 8$

$\tau = 8.2$

$\tau = 8.5$

Simulating Fractional Delays in MATLAB

- Code on previous slide was create with DSP toolbox

```
tau = [0,8,8.2,8.5];  % Delays in fractions of a sample
```

```
% Create a fractional delay object from the DSP toolbox
% We select the Farrow interpolation, which is a fast
% and accurate method.  It is important to select the options
% correctly
dly = dsp.VariableFractionalDelay(...
    'InterpolationMethod', 'Farrow','FilterLength',8,...
    'FarrowSmallDelayAction','Use off-centered kernel');
```

```
% Create delays of the sequence
y = dly.step(x,tau);
```

Creates T x D matrix
Row i is delayed by $\tau(i)$

In-Class Problem: Fractional Delays on Constellations

```
% TODO:  Generate nb=1024 bits using the randi command.

% TODO:  Modulate the bits using QPSK with the qammod command
%
%     x = qammod(bits,...);
%
% Set the 'InputType' to 'bit' in the qammod command


% TODO:  Create a delay object
%    dly = dsp.VariableFractionalDelay(...)
% You can follow the demo

% TODO:  Shift the symbols x by delays
%    tau = 10,10.1 and 10.5 samples

% TODO:  Plot the received constellations
% Use the subplot command so the constellations for the three
% delays occur on different windows
```

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- Review of Up- and Downconversion
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- Doppler and Multi-Path Fading
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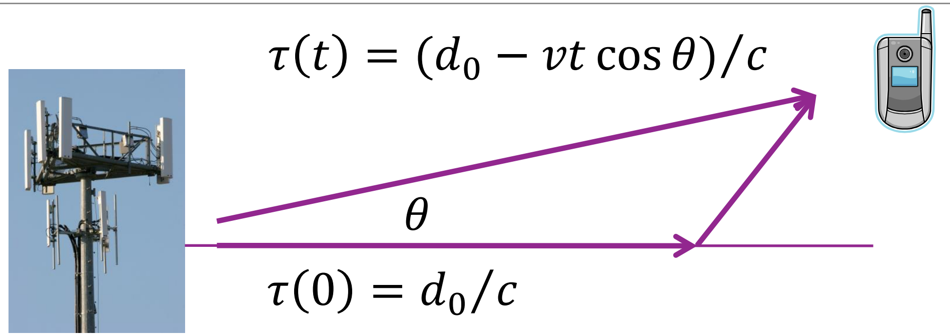
Doppler Shift

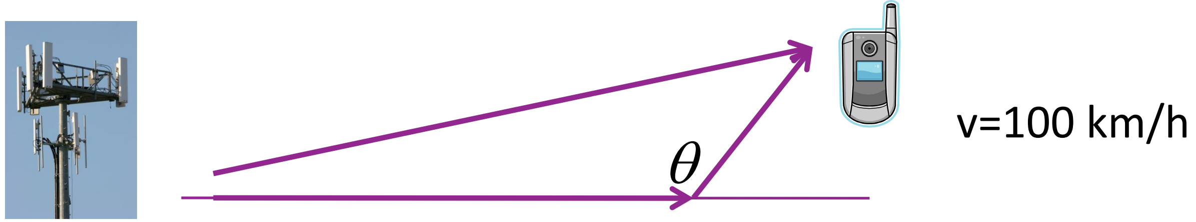


- With mobile velocity, propagation delay changes with time.
- In complex baseband signal:

$$r(t) = \alpha e^{-j\omega_c \tau(t)} u\big(t - \tau(t)\big) = \alpha e^{j2\pi(\Delta f t - d_0 f_c)} u(t - \tau(t))$$

- Velocity results in Doppler shift: $\Delta f = vt\cos\theta / c$
- Change in frequency, although not gain.

Sample Problem



- Suppose the carrier frequency is f_c=2.1GHz, and a car moves towards a base station at 100 km/h. What is the Doppler shift?
- Answer: v=100km/h= 27.7 m/s, c= $3(10)^8$ m/s, $\theta = 0$:

$$\Delta f = \frac{v f_c \cos\theta}{c} = \frac{(27.7)(2.1)(10)^9}{3(10)^8} \approx 194 \text{ Hz}$$

- If the angle is $\theta = 45$:

$$\Delta f = \frac{v f_c \cos\theta}{c} = \frac{(27.7)(2.1)(10)^9 \cos(45)}{3(10)^8} \approx 138 \text{ Hz}$$

Multipath Channel

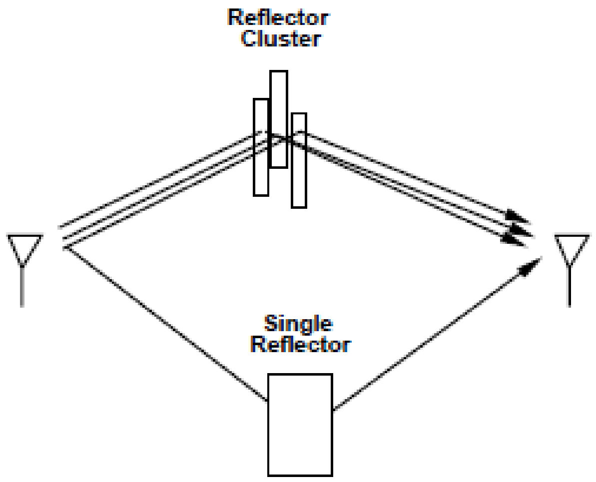


Figure 3.1: A Single Reflector and A Reflector Cluster.

- Most channel consists of many paths
  - Direct paths
  - Reflections, transmissions, diffraction, …
  - LOS and NLOS paths
- Each path has different
  - Delay
  - Phase
  - Gain

Baseband Model

- TX sends complex baseband $x(t)$
- RX receives complex baseband:

$$r(t) = \sum_{\ell=1}^{L} \alpha_\ell e^{-j\omega_c\tau_\ell + \phi_\ell} x(t - \tau_\ell)$$
$$= \sum_{\ell=1}^{L} g_\ell e^{-j\omega_c\tau_\ell} x(t - \tau_\ell)$$

  - L paths
  - Gain and phase: α_ℓ, ϕ_ℓ
  - Complex gain: $g_\ell = \alpha_\ell e^{j\phi_\ell}$
  - Delay: τ_ℓ
  - Doppler: $\omega_\ell = 2\pi f_{max} \cos\theta_\ell$

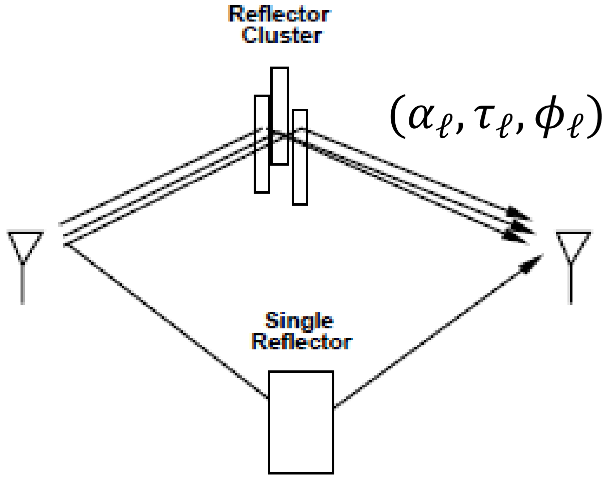


Figure 3.1: A Single Reflector and A Reflector Cluster.

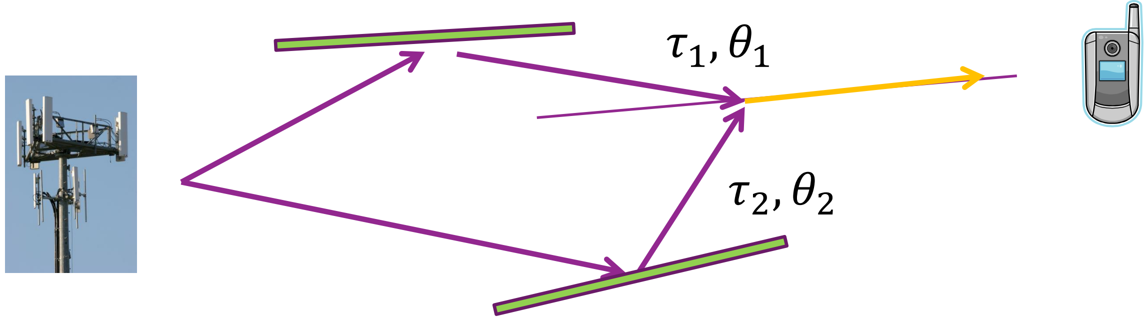
Time-Varying Frequency Response

- Multipath channel: $y(t) = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t} x(t-\tau_\ell)$
- Consider exponential input: $x(t) = e^{j\omega t}$
- Output is: $y(t) = H(t,\omega)x(t)$
- Time-varying frequency response

$$H(t,\omega) = \sum_{\ell=1}^{L} g_\ell e^{j(\omega_\ell t - \omega\tau_\ell)}$$

- May also write: $H(t,f) = H(t,2\pi f)$

Example with Two Paths



- To simplify understanding, consider two path model

$$r(t) = h_1 e^{j\omega_1 t} u(t - \tau_1) + h_2 e^{j\omega_2 t} u(t - \tau_2)$$

- Time-varying response:

$$H(t, \omega) = h_1 e^{j(\omega_1 t - \omega \tau_1)} + h_2 e^{j(\omega_2 t - \omega \tau_2)}$$

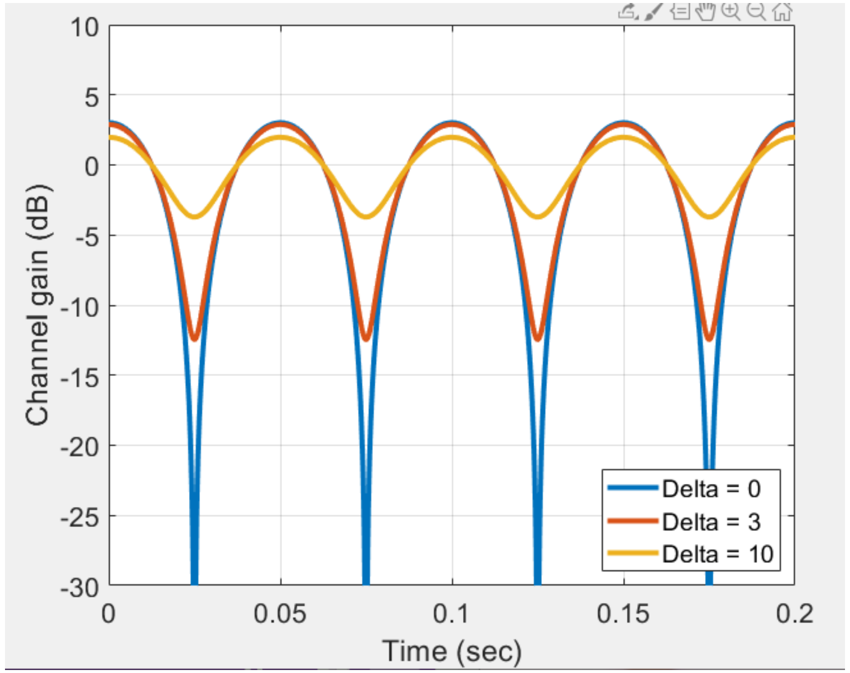
- Power gain:

$$P(t, \omega) = |H(t, \omega)|^2 = \left| h_1 e^{j(\omega_1 t - \omega \tau_1)} + h_2 e^{j(\omega_2 t - \omega \tau_2)} \right|^2$$

Variation in Time

$$P(t,\omega_0) = \left|h_1 e^{j(\omega_1 t+\phi_1)} + h_2 e^{j(\omega_2 t+\phi_2)}\right|^2$$

- Fixed frequency ω_0
- Look at time variations $P(t,\omega_0)$
- Rate of variation depends on Doppler spread:
  $$\Delta f = f_{max}(\cos\theta_1 - \cos\theta_2)$$
- Size of variation depends on spread of gains:
  - Avg: $|h_1|^2 + |h_2|^2$
  - Min: $(|h_1| - |h_2|)^2$: Destructive interference
  - Max: $(|h_1| + |h_2|)^2$: : Constructive interference
- With equal path gains, there are nulls



Plot shows $f_{\max}$ =10 Hz,
$\theta_1 = 0, \theta_2 = 180,$
$h_2 = 10^{-0.05\Delta} h_2,\ |h_1|^2 + |h_2|^2 = 1$

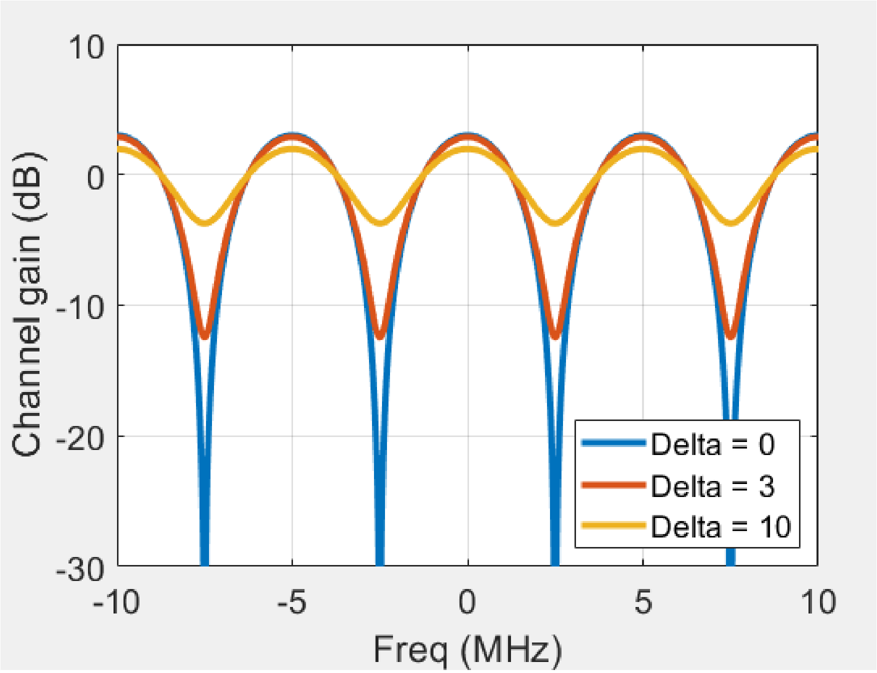
Variation in Frequency

$$P(t_0, \omega) = \left|h_1 e^{j(\omega\tau_1+\phi_1)} + h_2 e^{j(\omega\tau_2+\phi_2)}\right|^2$$

- Fixed frequency t_0
- Look at time variations $P(t, \omega_0)$
- Period of variation depends on delay spread:

$$\Delta f = \frac{1}{\tau_2 - \tau_1}$$

- Size of variation depends on spread of gains:
  - Avg: $|h_1|^2 + |h_2|^2$
  - Min: $(|h_1| - |h_2|)^2$
  - Max: $(|h_1| + |h_2|)^2$

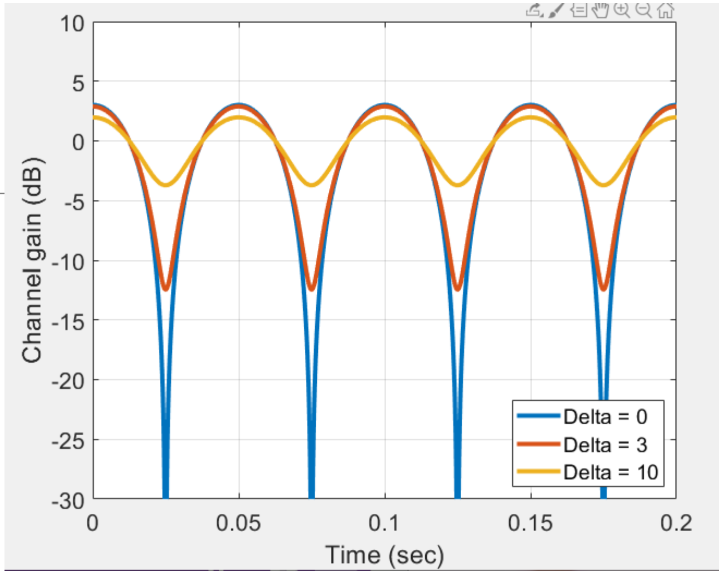


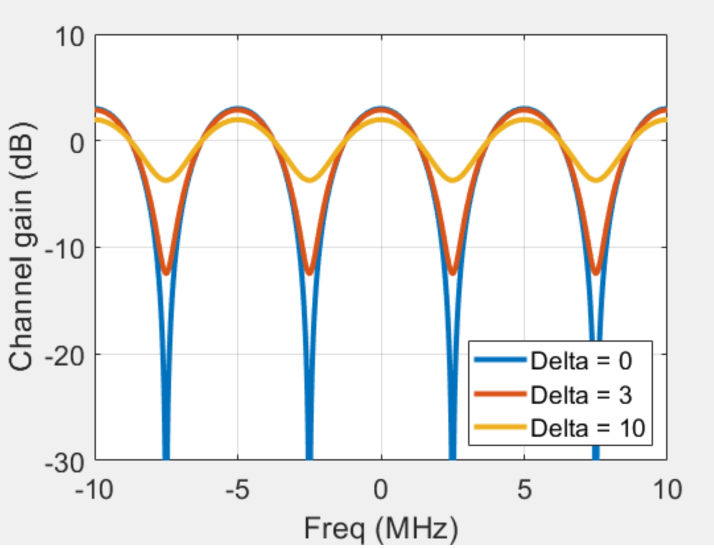
Plot shows
$\tau_1 = 0, \tau_2 = 200$ ns,
$h_2 = 10^{-0.05\Delta} h_2, \; |h_1|^2 + |h_2|^2 = 1$

Fading

- Over time and frequency, paths can either
  - Constructively interfere ⇒ Peaks
  - Destructively interfere ⇒ Nulls
- Process is called fading
  - Intermittent channel quality

- One of the most significant challenges in wireless
- Later, we will discuss how to overcome fading





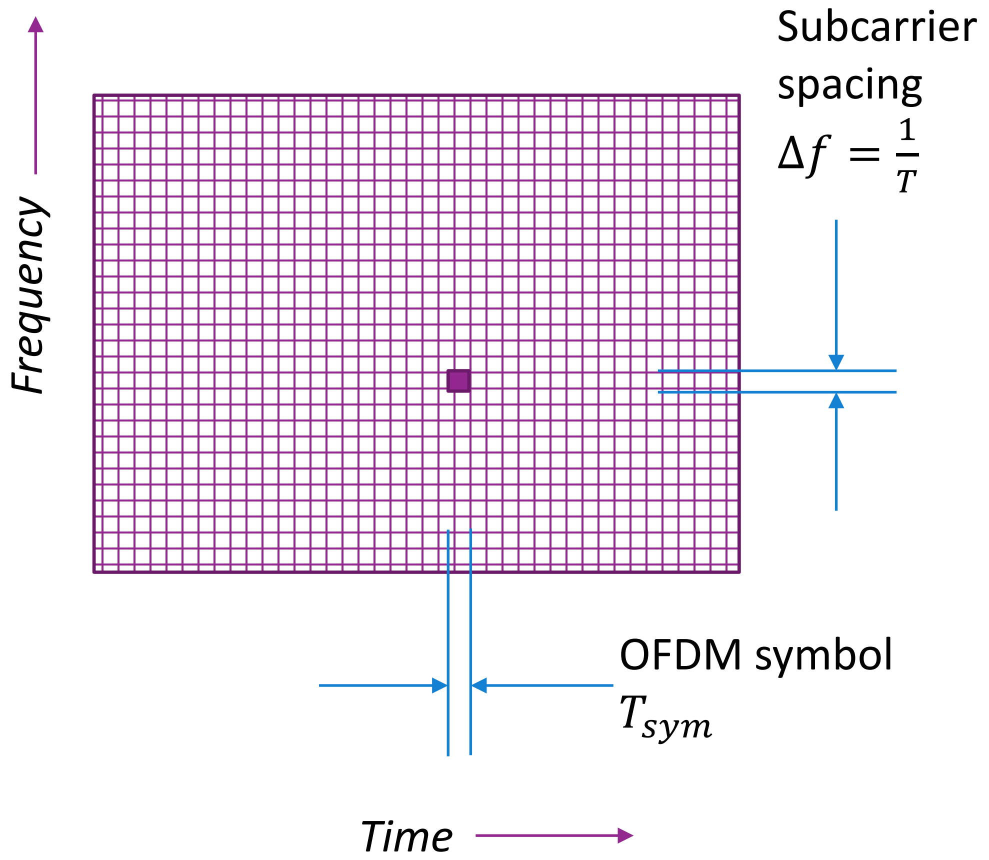
Narrowband Assumption

- For two path model:

$$P(t_0, \omega) = \left| h_1 e^{j(\tau_1 \omega + \phi_1)} + h_2 e^{j(\tau_2 \omega + \phi_2)} \right|^2$$

- Period of variation in f = $|\tau_1 - \tau_2|$
- Suppose that $u(t)$ has bandwidth W
- Narrowband assumption valid when $|\tau_1 - \tau_2| \ll W^{-1}$
- $|\tau_1 - \tau_2|$ is the delay spread
- Represents max difference in path lengths

OFDM Time-Frequency Grid



- OFDM modulation: Widely-used method
  - 4G and 5G cellular systems
  - Many 802.11 standards
- Divide channel into sub-carriers and OFDM symbols
  - Resource element: One time-frequency point
- Data is transmitted is an array: $X[n,k]$
  - $k =$ OFDM symbol index
  - n = subcarrier index
  - One complex value per RE.
  - Called a modulation symbol
- See digital communication class
  - We will also review again when we discuss equalization

OFDM Channel with Fading

- OFDM channel acts as multiplication:
  Under normal operation (delay spread is contained in CP):

$$Y[k,n] = H[k,n] \;\; X[k,n]$$

RX symbols — Channel — TX symbols

- OFDM channel gains can be computed from the multi-path components

$$H[k,n] = \sum_{\ell=1}^{L} \sqrt{E_\ell} e^{-2\pi j \, (Tkf_\ell + Sn\tau_\ell + \phi_\ell)}$$

  - T = OFDM symbol time, S = sub-carrier spacing
  - For each path: f_ℓ =Doppler shift, τ_ℓ =Delay, ϕ_ℓ = phase of path, E_ℓ = path received energy

Summary

- Doppler to a single path causes a phase rotation
  - Gain is constant
- With multiple paths, gain varies
  - Constructive and destructive interference of paths
- Described by a time-varying frequency response $H(t, f)$
  - Variations is time due to Doppler spread
  - Variations in frequency due to delay spread

In-Class Exercise: OFDM Channel Response

```
scs = 120e3;  % sub-carrier spacing
nsc = 12*60;  % number of sub-carriers
tsym = 1e-3/14/8;        % OFDM symbol period
nsym = 1000;  % number of symbols to plot

% Channel parameters
fc = 73e9;  % carrier frequency
v = 10;      % RX velocity in m/s
dly = [0,20,50]'*1e-9;   % Delay in sec of the paths
theta = [0,pi/4,pi]';    % Path AoA relative to motion
gaindB = [0,-3,-5]';     % gain of each path in dB

% Random initial phase of the gains
npath = length(dly);
phi = rand(npath,1)*2*pi;

% TODO:  Compute the Doppler shift of each path


% TODO:  Compute the OFDM channel H(k,n)


% TODO:  Plot the power 10*log10 |H(k,n)|^2.
% Use the imagesc function. Label the axes across
% time in ms and frequency in MHz
```

Outline

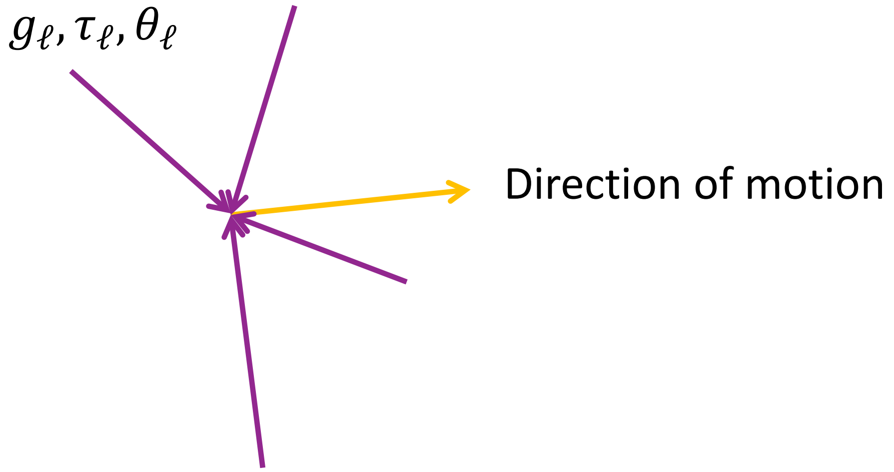
- Review of Up- and Downconversion
- Review of TX and RX Sampling
- Doppler and Multi-Path Fading
- Statistical Descriptions of Fading

Random Path Statistical Model

- RX signal has many random, independent paths
- Time-varying frequency response:

$$h(t,f) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell e^{2\pi i (t f_d \cos\theta_\ell + f\tau_\ell)}$$

  - Assume (g_ℓ, θ_ℓ) i.i.d.
  - Path gains: g_ℓ are zero mean $\mathrm{E}|g_\ell|^2 = P$

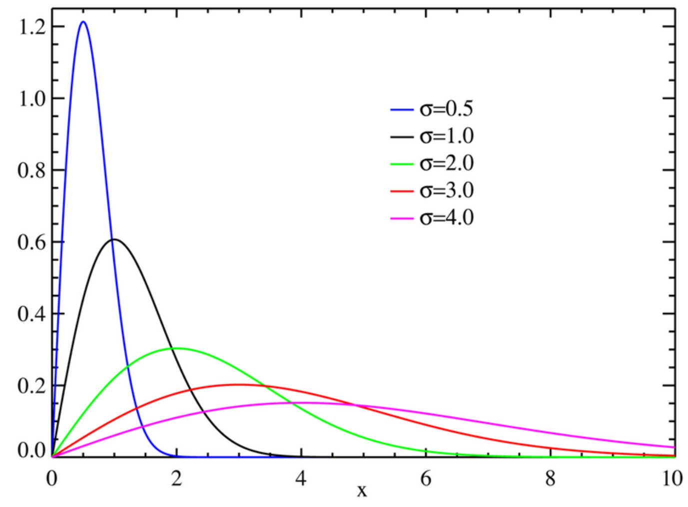


- By Central Limit Theorem, $h(t)$ is a complex Gaussian
  - $h(t,f) \sim CN(0,P)$
  - Independent real and imaginary components
  - Variance $P/2$ for real and imaginary components

Rayleigh Distribution

- $h \sim CN(0, P)$ complex Gaussian
- $R = |h|$ magnitude
- Represents amplitude gain
- Has Rayleigh distribution:
  - PDF: $p(r) = \frac{2r}{P} e^{-r^2/P}$
  - CDF: $\mathrm{P}(R \leq r) = 1 - e^{-r^2/P}$
  - Second moment: $ER^2 = P$

Probability distribution



Exponential Distribution

- Consider Rayleigh fading complex gain $h \sim CN(0, G_{avg})$
- Magnitude $R = |h|$ is Rayleigh

$$P(R \geq r) = e^{-r^2/G_{avg}}$$

- Instantaneous gain $G = |h|^2$ has exponential distribution

$$P(G \geq g) = P(R \geq \sqrt{g}) = e^{-g/G_{avg}}$$

  - Average gain is $E(G) = E|h|^2 = G_{avg}$
- For channel, G represent power gain (in linear scale)
  - $y = hx \Rightarrow \frac{|y|^2}{|x|^2} = G$

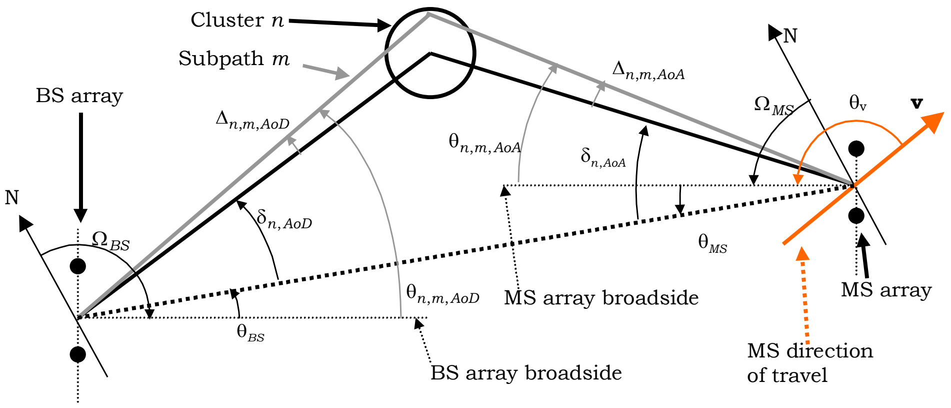
Example Calculation

- Suppose the channel experiences Rayleigh fading.
- What is probability gain will be 15 dB below the average?
  - Called a 15 dB fade.
- Answer:
  - Gain is 15 dB below average when $G \leq 10^{-0.1(15)} E(G)$
  - From exponential distribution:

$$P\big(G \leq \beta E(G)\big) = 1 - e^{-\beta E(G)/E(G)} = 1 - e^{-\beta}$$

  - For small β, $P\big(G \leq \beta E(G)\big) \approx \beta$
  - For 15 dB fade, $\beta = 10^{-0.1(15)} \approx 0.032.$

Winner-3GPP-Spatial Cluster Model



From 3GPP SCM-132

- Paths arrive in clusters.
- Clusters have subpaths (also called rays)
- Each cluster has:
  - Center angle and a statistical model for the delay and angular spread

Jakes Model

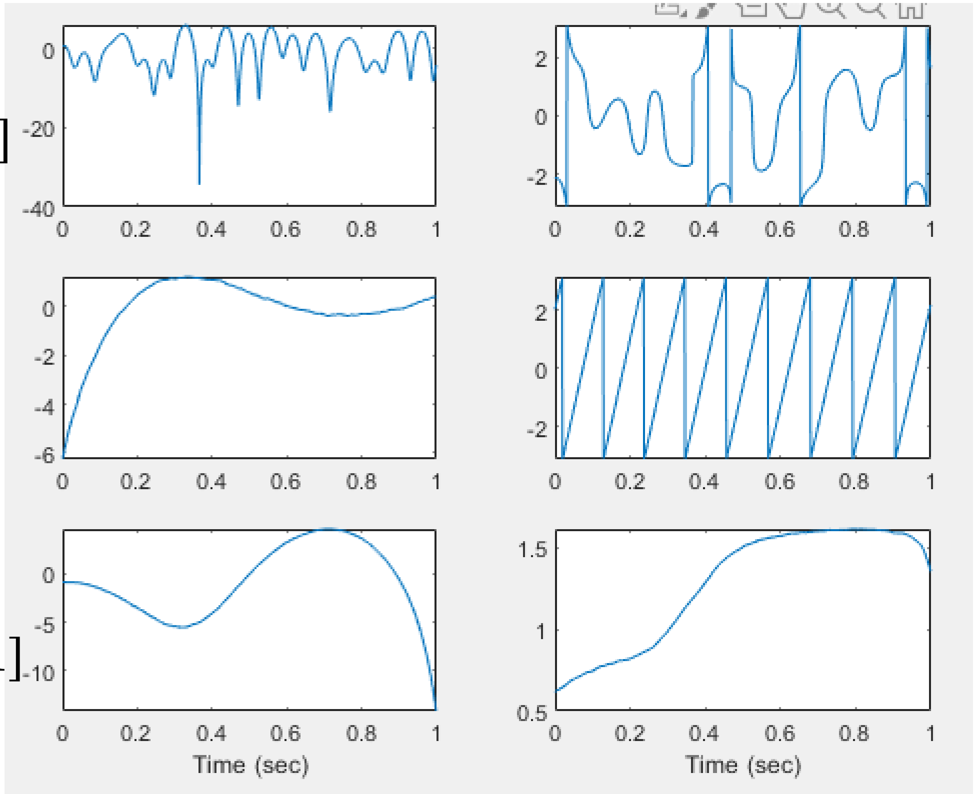
- Many widely-used statistical models in practice
- Some specify models with no delay spread
  - Angular spread only
  - Creates a time-varying gain $h(t)$
  - No variation in delay
  - Use one of these models per cluster
- Jakes model:
  - Angles uniform from $[0,2\pi]$
- Asymmetric Jakes:
  - $\theta \in [\theta_1, \theta_2]$ uniform
- Angular spread:
  - Arises from diffuse reflection

Gain (dB) | Angle (rads)

Jakes
Angles unif $[0,2\pi]$

Asym Jakes
$\cos\theta \in [0.9,1]$

Asym Jakes
$\cos\theta \in [-0.1,0.1]$



Fading Models in MATLAB

- Comm Toolbox:
  - Efficient, general fading models
- Create a comm.RayleighChannel object
- Run the channel to get:
  - Output and gain

```
% Create Doppler models
nmod = 3;
dopMod = cell(nmod,1);
dopMod{1} = doppler('Jakes');
dopMod{2} = doppler('Asymmetric Jakes', [0.9 1]);
dopMod{3} = doppler('Asymmetric Jakes', [-0.1 0.1]);

% Simulate the channel gains for each model
for i = (1:nmod)
    chan = comm.RayleighChannel(...
        'SampleRate', fsym, 'AveragePathGains', 0, ...
        'MaximumDopplerShift', fdmax,...
        'DopplerSpectrum', dopMod{i}, ...
        'PathGainsOutputPort', true);

    [y, gain] = chan.step(x);
```

Doppler Spectra

- Consider statistical model:

$$h(t,f) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell e^{2\pi i(tf_d \cos\theta_\ell + f\tau_\ell)}$$

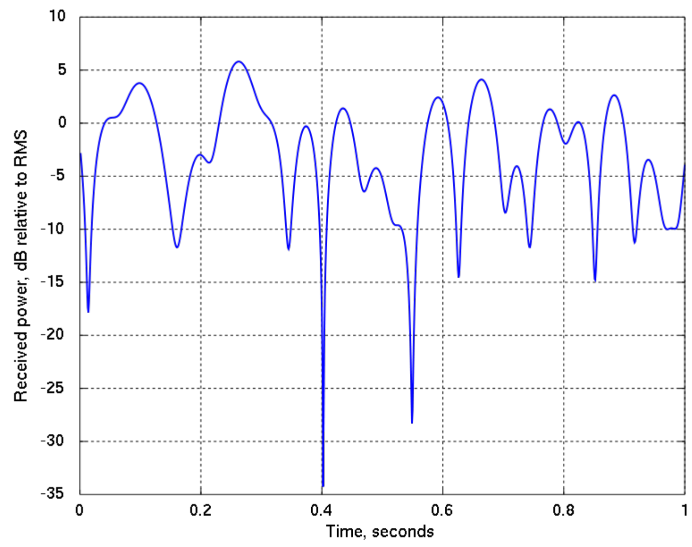
  - Paths are i.i.d. and g_ℓ are zero mean
- In limit of large L, $h(t,f)$ is a Gaussian random process
- Auto-correlation:

$$R(\delta t, \delta f) = E[h(t,f)h^*(t+\delta t, f+\delta f)] = P\, E\left\{e^{2\pi i(\delta t f_d \cos\theta_\ell + \delta f \tau_\ell)}\right\}$$

- Describes how correlated the process is over time and frequency

Coherence Time and Frequency

- Consider time varying freq response $H(t, f)$
- Coherence time:
  - Max interval Δt where $H(t, f) \approx H(t + \Delta t, f)$
  - How fast channel changes in time
  - Related to Doppler spread
- Coherence bandwidth
  - Max interval Δf where $H(t, f) \approx H(t, f + +\Delta f)$
  - How fast channel changes in frequency
  - Related to delay spread
- Critical for many procedures:
  - Channel estimation, tracking, coding, ARQ, …
  - More on this later



Realization of a Jakes process with $1/f_{max}$ = 0.1 sec

Fading at Different Scales

Source of variation	Mathematical model	Typical spatial coherence	Typical temporal coherence
Small-scale fading from multi-path fading	Rayleigh or Rician distribution	~ 1 wavelength	15 ms (v=10m/s, fc=2GHz)
Large-scale fading from variations in shadowing	Lognormal distribution	10 to 100 m	1 to 10 sec
Path loss variations	Path loss exponent	100 m or larger	10 sec

- Different fading processes and variations occur at much different time / space scales
- Methods to combat these are different

Time Scales Illustrated

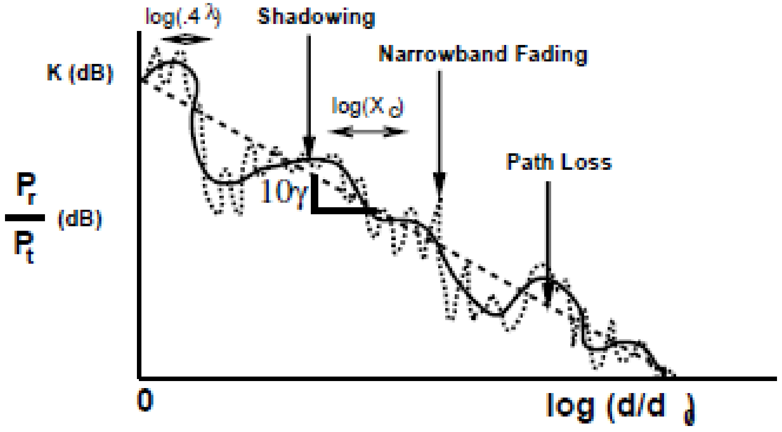


Figure 3.8: Combined Path Loss, Shadowing, and Narrowband Fading.