Unit 8. Multiple Antennas and Beamforming

EL-GY 6023. WIRELESS COMMUNICATIONS

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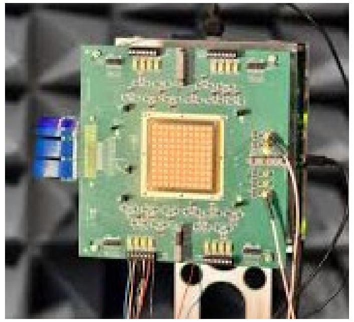
Outline

- Antenna Arrays and the Spatial Signature
- Receive Beamforming and SNR Gain with a Single Path
- Array Factor
- Transmit Beamforming with a Single Path
- Multipath and MIMO Channels
- Linear Algebra and SVD Review
- Beamforming Gains in Multipath Channels
- Adding Element Gains and Normalizing Spatial Signatures for Mutual Coupling

Antenna Arrays

- Antenna arrays: Structure with multiple antennas
  - At TX and/or RX
  - Key to 5G mmWave and massive MIMO
- Two key benefits
- Beamforming: This lecture
  - Concentrate power in particular directions
  - Increases SNR and may enable spatial diversity
  - Requires arrays at *either* TX or RX
- Spatial multiplexing: Later
  - Enables transmission in multiple virtual paths
  - Increases degrees of freedom
  - Requires multiple antennas at *both* TX and RX



IBM 28 GHz array
32 element dual polarized array
Sadhu et al, ISSCC 2017

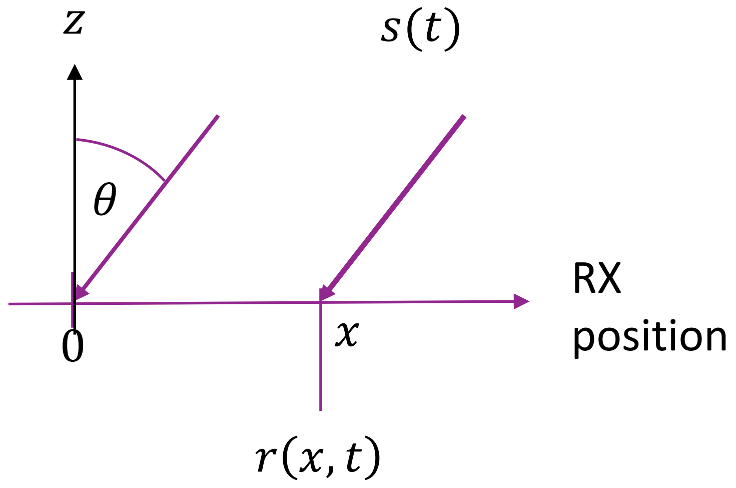
Aurora C-Band Massive MIMO array
64 elements, 5-6 GHz
https://www.taoglas.com/

Multiple Receive Antennas

- Single Input Multiple Output
  - One TX antenna
  - M RX antennas
- Transmit a scalar signal $x(t)$
- Receive a vector of signals:
  - $\boldsymbol{r}(t) = \left(r_1(t), \ldots, r_M(t)\right)^T$
- What is the channel from $x(t)$ to $\boldsymbol{r}(t)$?
- Want channel in complex baseband

Channel vs. Position

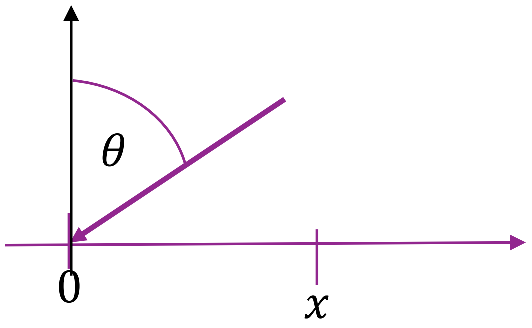
- Consider single path channel that arrives at origin with:
  - Delay τ_0, complex gain g_0, AoA of θ relative to z-axis
- Transmit signal $s(t)$
- Look at RX signal $r(x,t)$ as a function of position x
- Assume RX position, x, is close to origin
  - $B|x| \ll f_c\lambda,\ B =$ bandwidth of $s(t)$
- Phase rotation with displacement:
  - Baseband response at x is (proof on next slide):



$$r(x,t) \approx e^{2\pi j x \sin\theta/\lambda} \; g_0 s(t-\tau_0)$$

Phase rotation with x — Response at $x = 0$

Proof of Phase Rotation with Displacement



RX position

- Delay of path at x is: $\tau(x) = \tau_0 - \frac{\mathrm{x}\sin\theta}{c}$
- Hence there is an additional delay: $-\frac{\mathrm{x}\sin\theta}{c}$
- Baseband response at x:

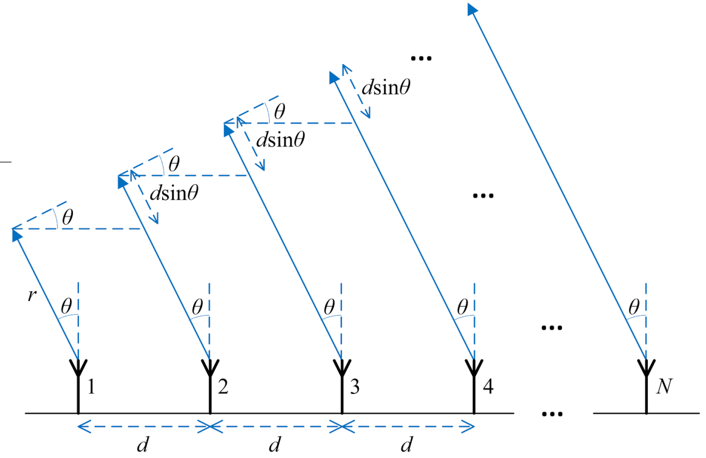
$$r(x,t) = g_0 e^{2\pi j x \sin\theta/\lambda} s(t-\tau(x))$$

- Also, $s\big(t-\tau(x)\big) \approx s(t-\tau_0)$ if $\mathrm{B}|\tau(x)-\tau_0| \ll 1$
- But, by assumption of small displacement:

$$\mathrm{B}|\tau(x)-\tau_0| \le \frac{B|x|}{c} = \frac{B|x|}{\lambda f_c} \ll 1$$

- Hence, $r(x,t) \approx g_0 e^{2\pi j x \sin\theta/\lambda} s(t-\tau_0)$

Response for a ULA



- ❑ Uniform Linear array (ULA)
  - M antenna positions spaced d apart
- ❑ Transmit signal $s(t)$
  - Channel single path with AoA θ, complex gain g
- ❑ Response at position: $r_m(t) = g_0 e^{2\pi j(n-1)d \sin\ \theta/\lambda} s(t-\tau_0)$
- ❑ In vector notation, we can write $\boldsymbol{r}(t) = \boldsymbol{h} s(t-\tau_0)$
  - $\boldsymbol{h}$ is the channel vector

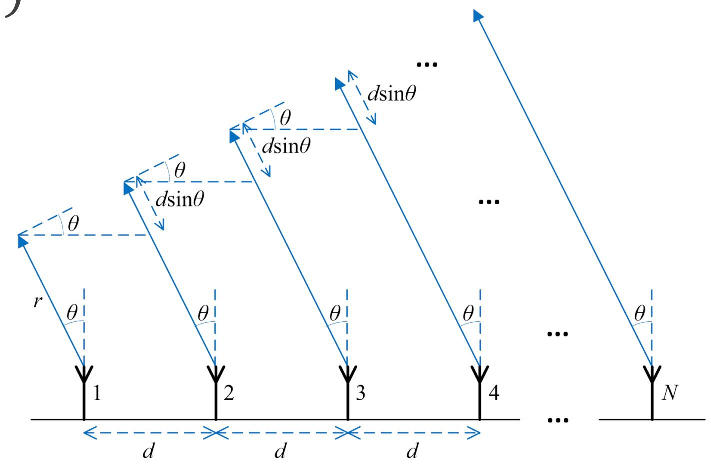
$$\boldsymbol{h} = g \begin{bmatrix} e^{2\pi j 0 d \sin\theta/\lambda} \\ \vdots \\ e^{2\pi j (M-1) d \sin\theta/\lambda} \end{bmatrix} = g\boldsymbol{u}(\theta)$$

Response Decomposition

- For a single path channel, the channel vector has two components:

$$r(t) = \boldsymbol{h}(\theta)s(t-\tau_0), \qquad \boldsymbol{h}(\theta) = g\boldsymbol{u}(\theta)$$

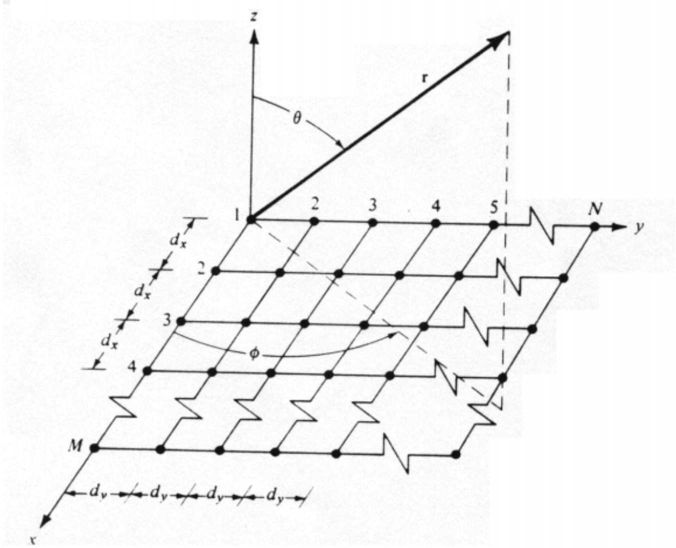
- Scalar channel gain, g
  - Complex channel gain at a reference position in the array



- Vector spatial signature, $\boldsymbol{u}(\theta)$
  - $\boldsymbol{u}(\theta) = \begin{bmatrix} e^{2\pi j 0 d \sin\theta/\lambda} \\ \vdots \\ e^{2\pi j (M-1) d \sin\theta/\lambda} \end{bmatrix}$
  - Vector of phase shifts from the reference
  - Also called the steering vector (reason for name will be clear later)

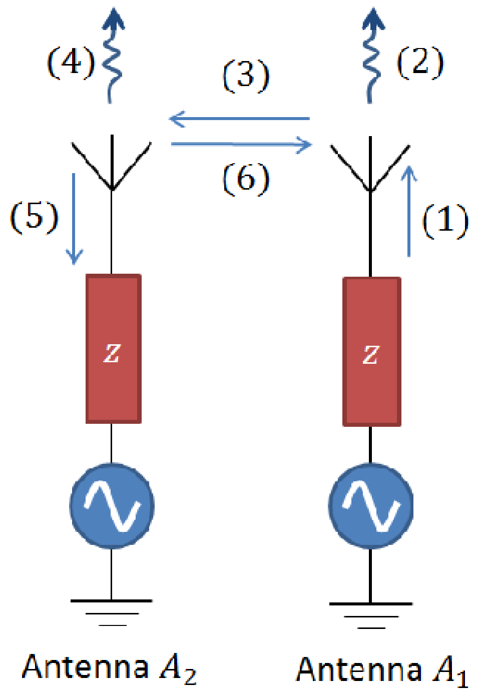
Array Response in 3D

- Many arrays place elements over 2D area
- Uniform rectangular array (URA):
  - $M \times N$ grid of elements
  - Spaced d_x and d_y
  - Also called uniform planar array (UPA)
- Incident angle $\Omega = (\phi, \theta)$
  - (Azimuth, elevation) or (azimuth, inclination)
- Spatial signature:
  - $u_{mn}(\Omega) =$ complex response to antenna (m, n)
  - $u_{mn}(\Omega) = \exp\left[\frac{2\pi i}{\lambda}\,(md_x \sin\theta \cos\phi + nd_y \sin\theta \sin\phi)\right]$



Mutual Coupling

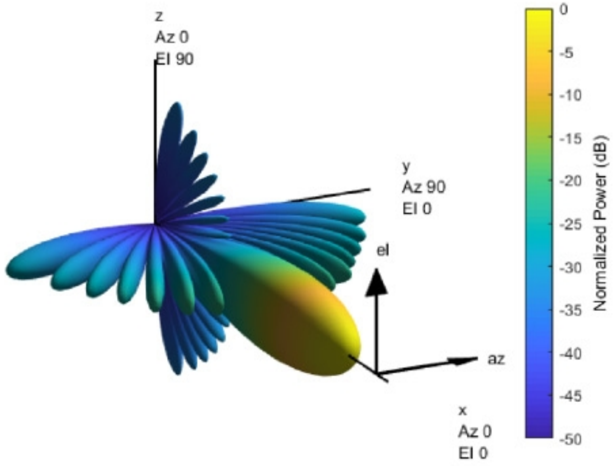
- The above formulas assume there is no mutual coupling
- Mutual coupling:
  - Signals on one antenna scatter to another antenna
  - Changes the antenna response
- Mutual coupling effect is typically large when:
  - Antennas are close
  - Or arrays are combined with highly directive elements
- We will show how to account for mutual coupling at the end of unit



Wang, Zhengzheng. "Complete tool for predicting the mutual coupling in non-uniform arrays of rectangular aperture radiators." (2017).

MATLAB Phased Array Toolbox

- Powerful toolbox
- Routines for:
  - Defining and visualizing arrays
  - Computing beam patterns
  - Beamforming
  - MIMO
  - Radar
  - …



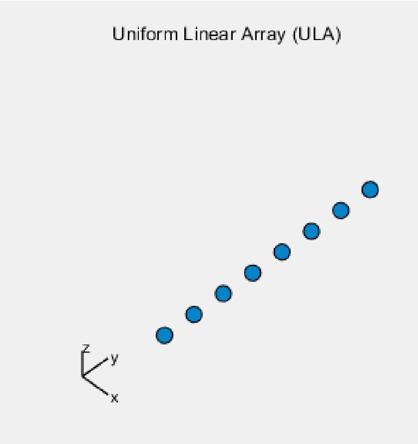
Example: Defining a ULA

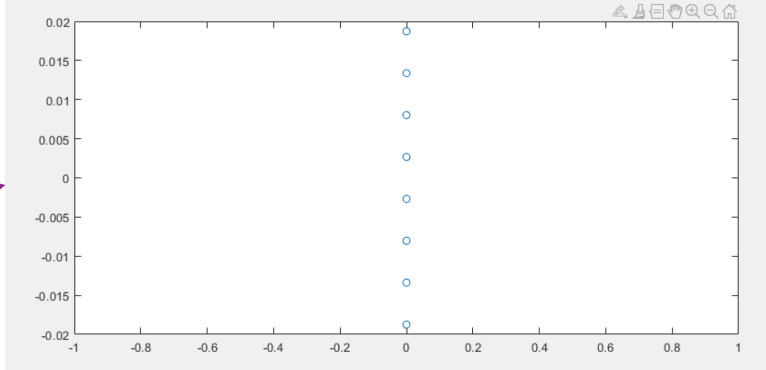
- ❑ Define and view the array
- ❑ Can display array:
  - Using viewArray command
  - Or, manually

```
%% Uniform Linear Array
% We first define a simple uniform linear array
fc = 28e9;              % frequency
lambda = physconst('LightSpeed')/fc;
dsep = 0.5*lambda;      % element spacing
nant = 8;               % Number of elements
arr = phased.ULA(nant,dsep);

% View the array
viewArray(ula,'Title','Uniform Linear Array (ULA)')
```

```
elemPos = arr.getElementPosition();
clf('reset');
plot(elemPos(1,:), elemPos(2,:), 'o');
```





Computing the Spatial Signature

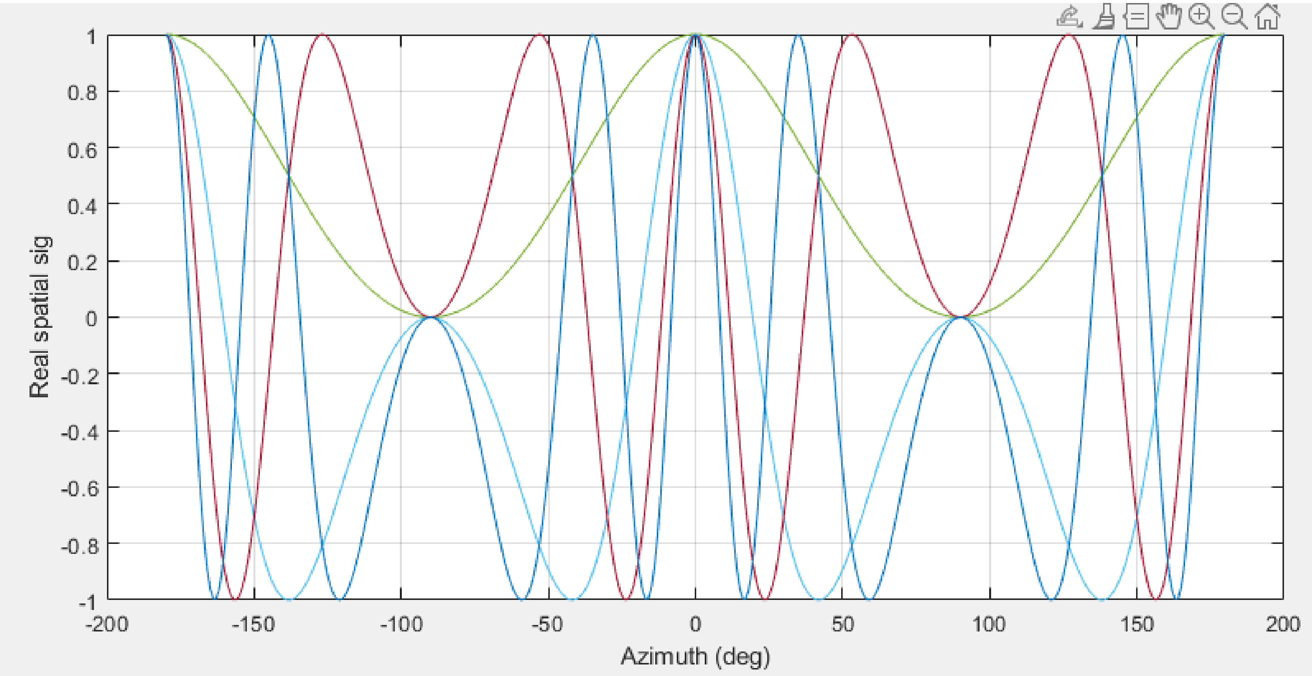
- Compute the spatial signature with the SteeringVector object

```
% Create a steering vector object
sv = phased.SteeringVector('SensorArray',arr);

% Angles to compute the SVs
npts = 361;
az = linspace(-180,180,npts);
el = zeros(1,npts);
ang = [az; el];

% Matrix of steering vectors
% This is an nant x npts matrix in this case
u = sv(fc, ang);

% Plot of the real components
plot(az, real(u)');
grid on;
xlabel('Azimuth (deg)')
ylabel('Real spatial sig');
```

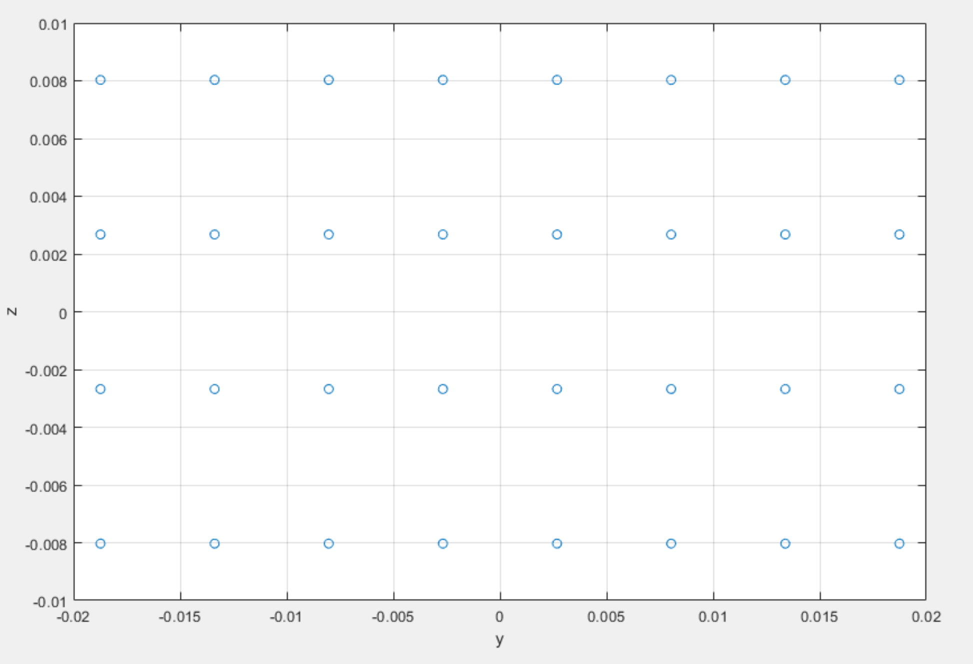


Example: Defining a URA

- Define and view the array
- Use the phased.URA class
- Can compute steering vector similarly

```
% Construct the array
nant = [4,8];
dsep = 0.5*lambda;
arr = phased.URA(nant,dsep,'ArrayNormal','x');

% Plot the array.
% You can also use, arr.viewArray()
elemPos = arr.getElementPosition();
clf('reset');
plot(elemPos(2,:), elemPos(3,:), 'o');
grid on;
xlabel('y');
ylabel('z');
```



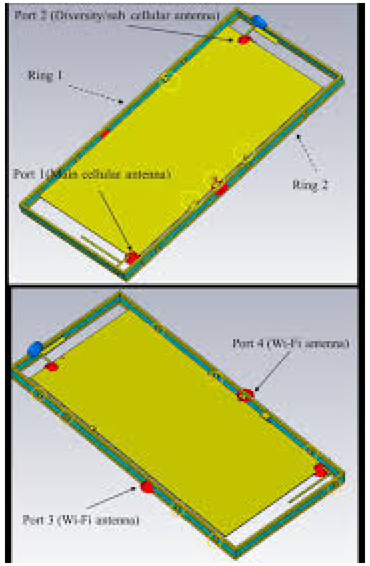
A 4 x 8 URA with normal axis aligned on x

Multiple Antennas in Commercial Systems

- Sub 6 GHz systems: Mostly 1 to 4 antennas on base stations or smart phones
- Form factor restricts larger number of antennas

WiFi Router
Linksys AC2200 with 4TX/RX

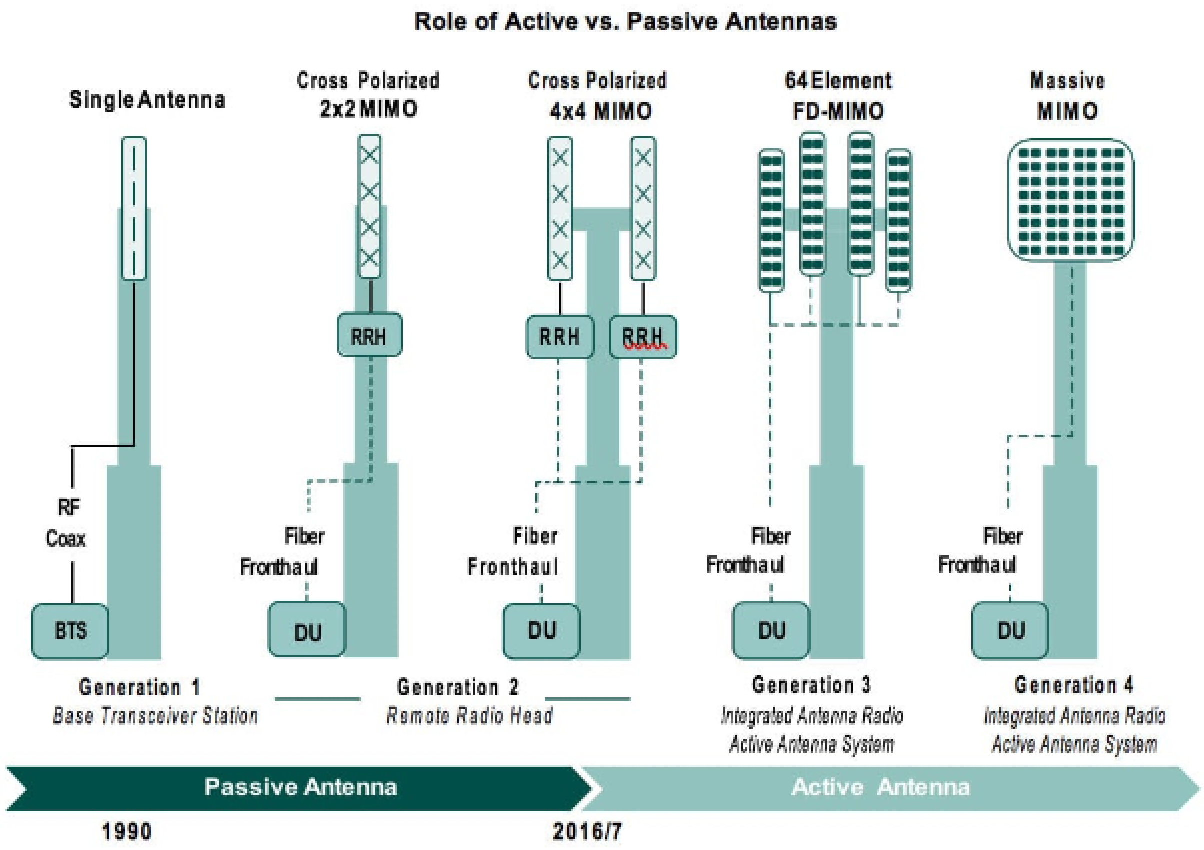
2x2 LTE base station antenna
Cros-polarization
16 dBi element gain, 90 deg sector
750x120x60mm



K. Zhao, S. Zhang, K. Ishimiya, Z. Ying and S. He, "Body-Insensitive Multimode MIMO Terminal Antenna of Double-Ring Structure," in *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 5, pp. 1925-1936, May 2015.

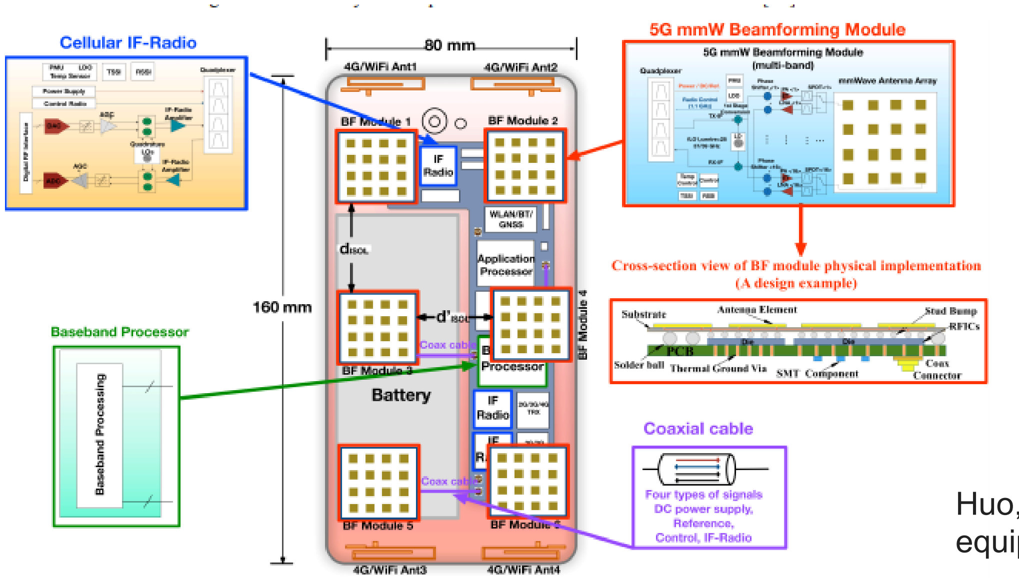
Massive MIMO

- Massive MIMO:
  - Many base station antennas
  - 64 to 128 in many systems today
- Significant capacity increase
  - Typically 8x by most estimates
- Use SDMA
  - Will discuss this later

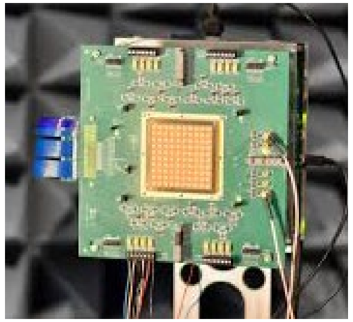


Beamforming and MmWave

- To compensate for high isotropic path loss, mmWave systems need large number of antennas
- 5G handsets: Multiple arrays with 4 to 8 antennas each
- 5G base stations: 64 to 256 elements



Huo, Yiming, et al. "Cellular and WiFi co-design for 5G user equipment." *2018 IEEE 5G World Forum (5GWF)*. IEEE, 2018.



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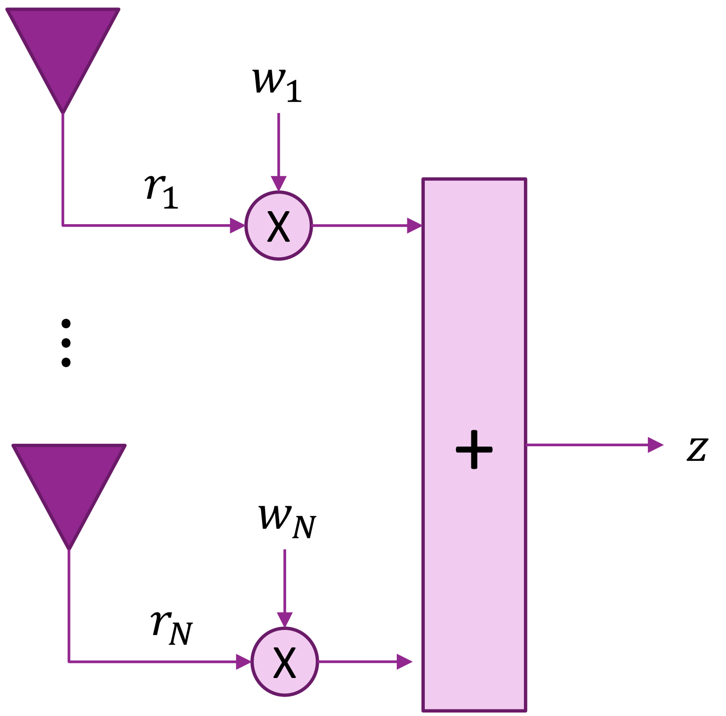
RX Beamforming

- Consider a general channel: $\boldsymbol{r} = \boldsymbol{h}x + \boldsymbol{n}$
  - 1 input, M outputs
- Beamforming: Take a linear combination of signals
  - $z = \boldsymbol{w}^T\boldsymbol{r} = \sum_j w_j r_j$
  - $\boldsymbol{w}$ is called beamforming vector for multiple antennas
- Creates effective SISO channel:

$$z = \boldsymbol{w}^T\boldsymbol{r} = (\boldsymbol{w}^T\boldsymbol{h})x + \boldsymbol{w}^T\boldsymbol{n} = \alpha x + v$$

  - 1 input x, 1 output symbol z
  - Gain: $\alpha = \boldsymbol{w}^T\boldsymbol{h}$
  - Noise: $v = \boldsymbol{w}^T\boldsymbol{n}$



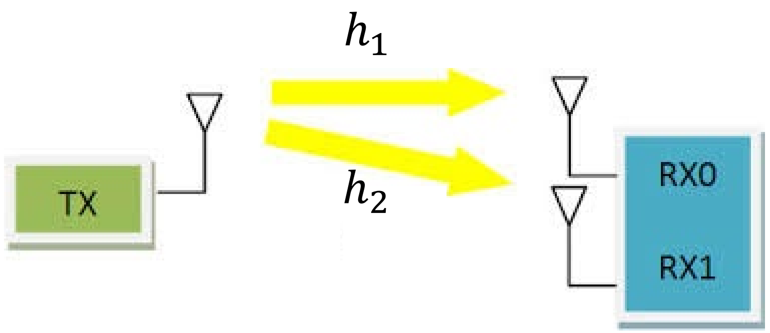
Conjugate Transpose Conventions

- For beamforming, we will use the following conventions
- Complex conjugate of a complex scalar $z = a + bi$ is denoted $\bar{z} = a - bi$
- Unless otherwise specified, vectors are column vectors: $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- Transpose: $\boldsymbol{x}^T = [x_1 \quad \cdots \quad x_n]$
- Conjugate transpose: $\boldsymbol{x}^* = [x_1^* \quad \cdots \quad x_n^*]$
- Elementwise conjugate: $\overline{\boldsymbol{x}} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$
  - Takes conjugate of each element but keeps $\boldsymbol{x}$ a column vector

Beamforming Analysis

- Linear combining: $z = \boldsymbol{w}^T\boldsymbol{r} = (\boldsymbol{w}^T\boldsymbol{h})x + \boldsymbol{w}^T\boldsymbol{n}$
  - Gain: $\alpha = \boldsymbol{w}^T\boldsymbol{h}$
  - Noise: $v = \boldsymbol{w}^T\boldsymbol{n}$
- Analysis: Let
  - $E_x = E|x|^2$ = average symbol energy
  - Assume noise $n_m \sim CN(0, N_0)$ (i.i.d. complex Gaussian noise)
- Then, after combining;
  - Signal energy = $|\boldsymbol{w}^T\boldsymbol{h}|^2 E_x$
  - Noise: v is Gaussian with $E|v|^2 = \|\boldsymbol{w}\|^2 N_0$
  - SNR is:

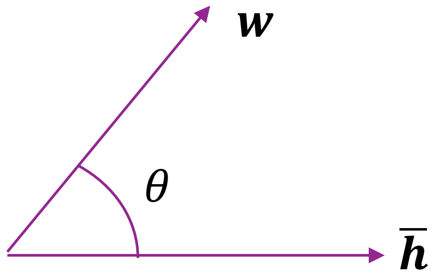
$$\gamma = \frac{|\boldsymbol{w}^T\boldsymbol{h}|^2 E_x}{\|\boldsymbol{w}\|^2 N_0}$$



Maximum Ratio Combining

- From previous slide: SNR is $\gamma = \dfrac{\left|\boldsymbol{w}^T\boldsymbol{h}\right|^2 E_x}{\|\boldsymbol{w}\|^2 N_0}$
- Maximum ratio combining: Select BF vector to maximize SNR: $\widehat{\boldsymbol{w}} = \arg\max_{\boldsymbol{w}} \dfrac{\left|\boldsymbol{w}^T\boldsymbol{h}\right|^2 E_x}{\|\boldsymbol{w}\|^2 N_0}$
- Theorem: The MRC weighting vector and maximum SNR is:

$$\widehat{\boldsymbol{w}} = c\bar{\boldsymbol{h}} \Rightarrow \gamma_{MRC} = \|\boldsymbol{h}\|^2 \frac{E_x}{N_0}$$



  - Any constant $c \neq 0$ can be used. Constant does not matter
  - Align BF vector with the conjugate of the channel
- Also called conjugate beamforming

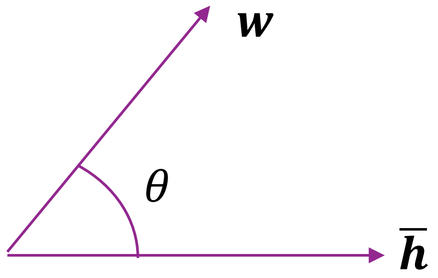
Proof of the MRC Solution

- We want to maximize $\hat{\boldsymbol{w}} = \arg\max_{\boldsymbol{w}} \frac{|\boldsymbol{w}^T\boldsymbol{h}|^2 E_x}{\|\boldsymbol{w}\|^2 N_0}$
- Write the inner product as:

Double conjugate

$$\bar{\boldsymbol{h}}^*\boldsymbol{w} = \sum w_i \bar{\bar{h}}_i = \sum w_i h_i = |\boldsymbol{w}^T\boldsymbol{h}|$$

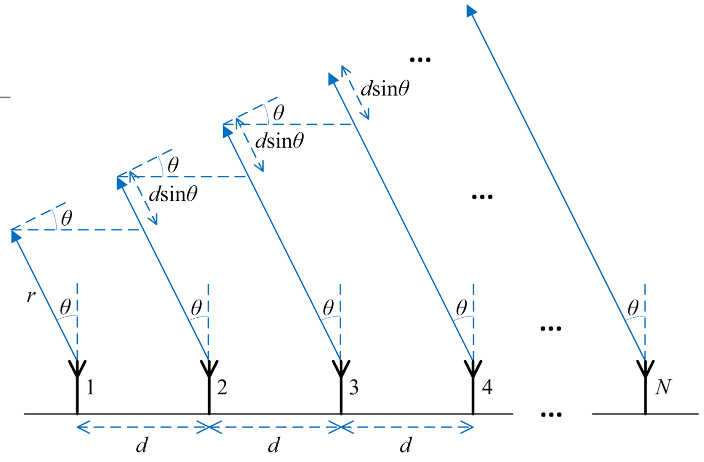
- Hence, we want to maximize $\hat{\boldsymbol{w}} = \arg\max_{\boldsymbol{w}} \frac{|\bar{\boldsymbol{h}}^*\boldsymbol{w}|^2 E_x}{\|\boldsymbol{w}\|^2 N_0}$
- From Cauchy-Schwartz: $\left|\bar{\boldsymbol{h}}^*\boldsymbol{w}\right|^2 = \|\boldsymbol{w}\|^2\left\|\bar{\boldsymbol{h}}\right\|^2 \cos\theta$
    - Hence, $\gamma = \left\|\bar{\boldsymbol{h}}\right\|^2 \frac{E_x}{N_0}\cos\theta = \|\boldsymbol{h}\|^2 \frac{E_x}{N_0}\cos\theta$
    - Maximized with $\cos\theta = 1 \Rightarrow \theta = 0$
- So, we take $\boldsymbol{w} = c\bar{\boldsymbol{h}}$



MRC Gain

- SNR with MRC: $\gamma_{MRC} = \|\boldsymbol{h}\|^2 \frac{E_x}{N_0}$
- SNR on channel i is: $\gamma_i = \frac{|h_i|^2 E_x}{N_0}$
- Average SNR is: $\gamma_{avg} = \frac{1}{M}\sum_{i=1}^{M} \gamma_i = \frac{1}{M}\sum_{i=1}^{M} |h_i|^2 \frac{E_x}{N_0} = \frac{1}{M}\|\boldsymbol{h}\|^2 \frac{E_x}{N_0}$
- MRC increases SNR by a factor of M relative to average per channel SNR
- Beamforming gain = $\frac{\gamma_{MRC}}{\gamma_{avg}} = M$
- Example: Suppose average SNR per antenna is 10 dB.
  - With $M = 16$ antennas and MRC, SNR $= 10 + 10\log_{10}(16) = 10 + 4(3) = 22$ dB
  - Gain increases significantly!
- Note: The gain assumes no mutual coupling.
  - Once antennas are close, the gain will no longer increase by M

Single Path Channel Case



- Consider special case of single path channel: $\boldsymbol{r} = g_0 \boldsymbol{u}(\Omega) x + \boldsymbol{n}$
  - Channel is $\boldsymbol{h} = g_0 \boldsymbol{u}(\Omega)$
- SNR per antenna (before beamforming):
  - $\gamma_0 = \frac{E_x |g_0|^2}{N_0} |u_m(\Omega)|^2 = \frac{E_x |g_0|^2}{N_0}$
  - Assume $u_m(\Omega)$ includes only phase shifts
- SNR after BF: $\gamma = \frac{\left|\boldsymbol{w}^T \boldsymbol{u}(\Omega)\right|^2}{\|\boldsymbol{w}\|^2} \gamma_0$
- MRC beamforming: $\widehat{\boldsymbol{w}} = c \overline{\boldsymbol{u}}(\Omega)$ and $\gamma = \|\boldsymbol{u}(\Omega)\|^2 \gamma_0 = M \gamma_0$
- Conclusions:
  - Optimal (MRC) beamforming vector is aligned to the conjugate of the spatial signature
  - Optimal SNR gain $= M$ (assuming no mutual coupling)
  - Linear gain with number of antennas

Example Problem

- Consider a system
  - TX power = 23 dBm with antenna directivity = 10 dBi
  - Free space path loss $d = 1000$ m
  - Sample rate = 400 Msym/s
  - Noise energy = -170 dBm/Hz (including NF)
  - RX antenna directivity = 5 dBi and 8 elements
- Find SNR per antenna and SNR with MRC
- Solution: We get a 9 dB gain!

```
SNR per  ant:      0.59
SNR with MRC:      9.62
```

```
% SNR per antenna
plomni = fspl(dist, lambda);
EsN0Ant = ptx - plomni - 10*log10(bw) - Enoise + dirtx + dirrx;

% SNR with MRC
EsN0MRC = EsN0 + 10*log10(nantrx);
```

In-Class Problem: Simple QPSK simulation

- Simulate QPSK transmission over a single path channel

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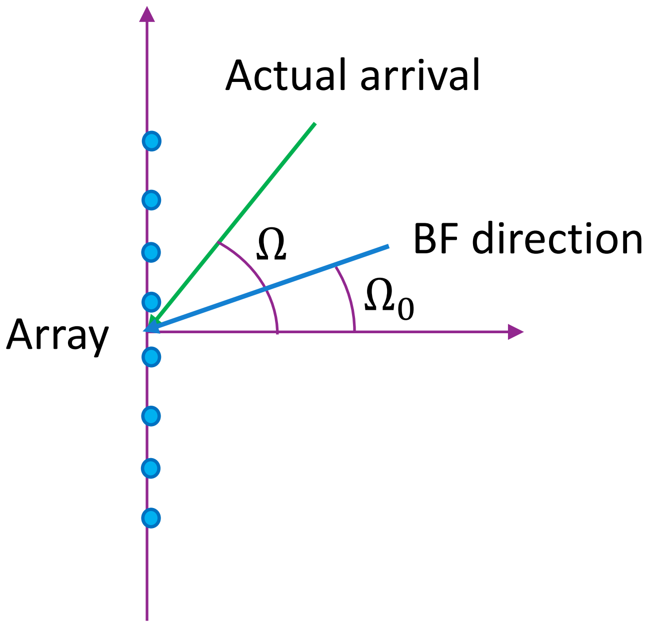
Array Factor

- Suppose RX aligns antenna for AoA $\Omega_0 = (\theta_0, \phi_0)$
- But signal arrives from AoA $\Omega = (\theta, \phi)$
- Define the (complex) array factor

$$AF(\Omega, \Omega_0) = \widehat{\boldsymbol{w}}^T(\Omega_0)\boldsymbol{u}(\Omega) = \frac{1}{\sqrt{M}}\boldsymbol{u}^*(\Omega_0)\boldsymbol{u}(\Omega)$$

  - Assume $||\widehat{w}|| = 1$
  - Indicates directional gain as a function of AoA θ
  - Dependence on θ_0 often omitted



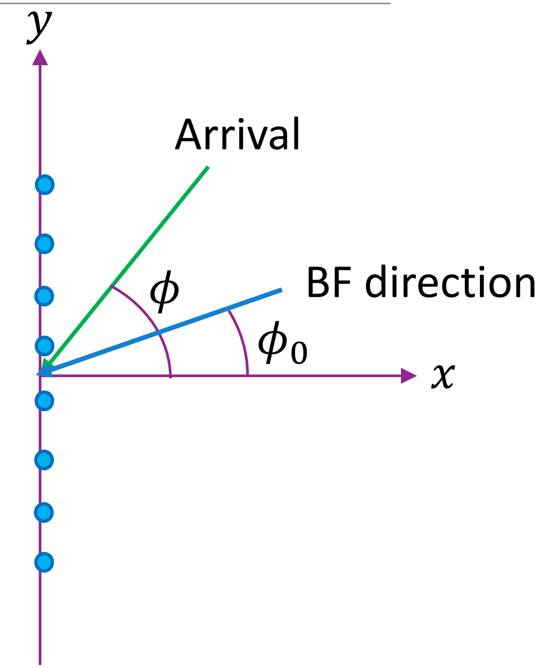
- SNR gain = $|AF(\Omega, \Omega_0)|^2$
  - Max value = M
  - Usually measured in dBi (dB relative to isotropic)
  - Also called the array response

Uniform Linear Array

- Spatial signature (for azimuth angle ϕ):
  - $\boldsymbol{u}(\phi) = \left[1, e^{j\beta}, \dots, e^{i(M-1)\beta}\right]^T, \ \beta = \frac{2\pi d \cos\phi}{\lambda}$
  - Note change from $\sin\theta$ to $\cos\phi$. (Array aligned on y-axis)
- Optimal BF vector for AoA ϕ_0
  - $\widehat{\boldsymbol{w}}(\phi_0) = \frac{1}{\sqrt{M}}\overline{\boldsymbol{u}}(\phi_0)$ (Note normalization)
- Array factor:

$$AF(\phi, \phi_0) = \frac{1}{\sqrt{M}}\boldsymbol{u}^*(\phi_0)\boldsymbol{u}(\phi) = \frac{e^{j(M-1)\gamma/2}}{\sqrt{M}}\frac{\sin(M\gamma/2)}{\sin(\gamma/2)},$$

  - $\gamma = \frac{2\pi d}{\lambda}(\cos\phi - \cos\phi_0),$
- Antenna gain: $|AF|^2 = \frac{\sin^2(M\gamma/2)}{M\sin^2(\gamma/2)}$

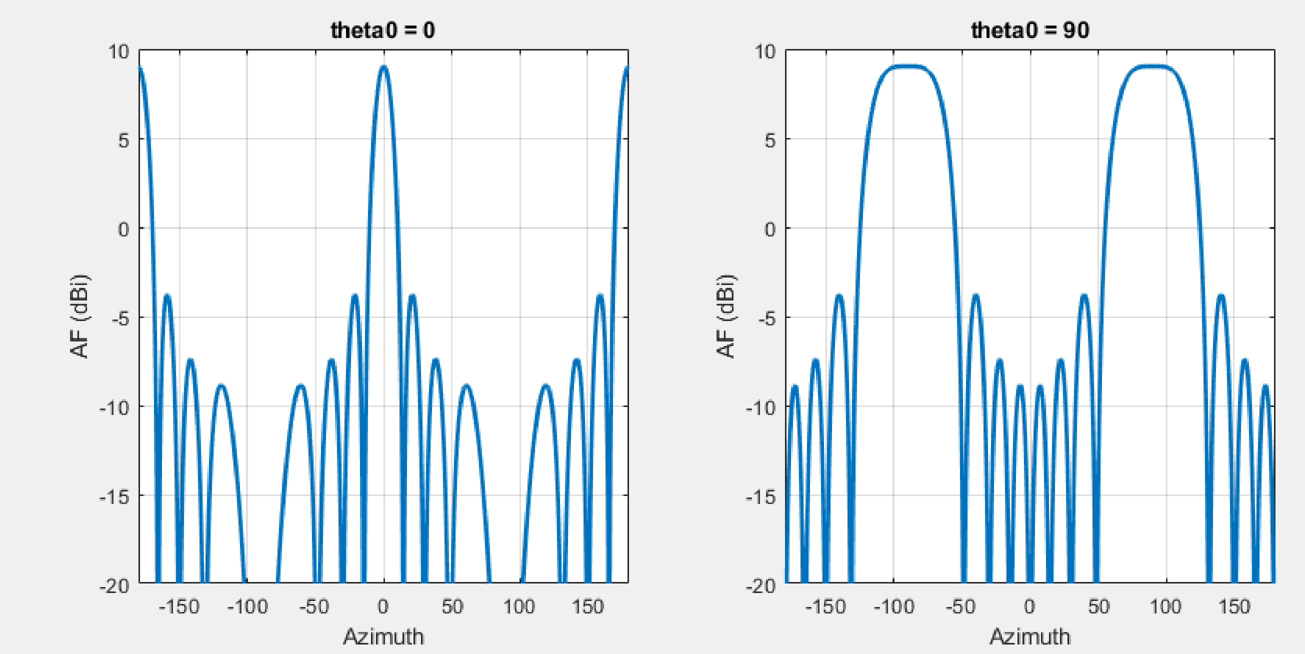


Antenna Gain for ULA

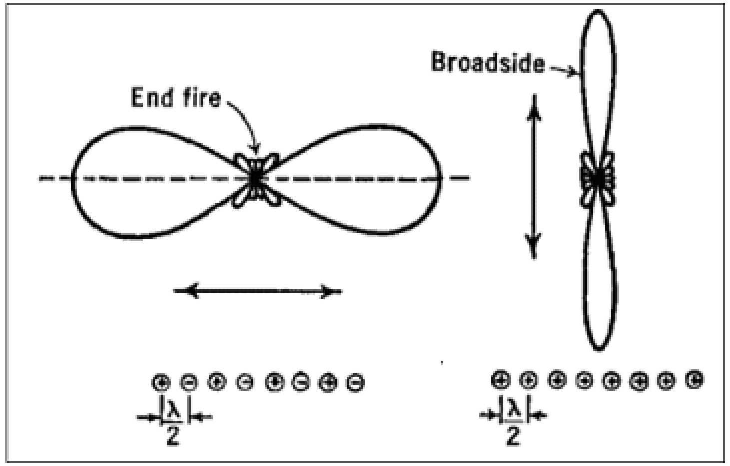
$d = \lambda/2\,, \qquad M = 8$

Broadside: $\theta_0 = 0$

Endfire: $\theta_0 = 90$



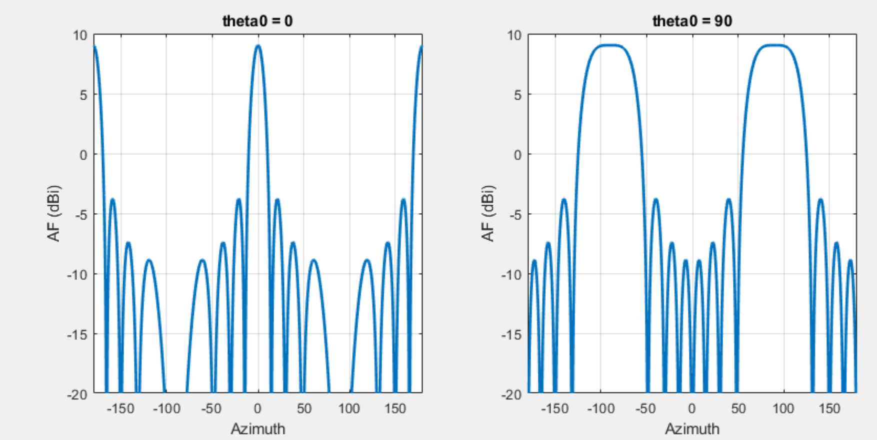
- Maximum gain of
- Note:
  - Endfire vs. broadside
  - Beamwidth $\propto\ 1/M$



Plotting the Array Factor

- Create a SteeringVector object
- Get steering vectors
- Compute inner products

```
% Create a steering vector object
sv = phased.SteeringVector('SensorArray',arr);

% Reference angles to plot the AF
azPlot = [0, 90];
nplot = length(azPlot);
```

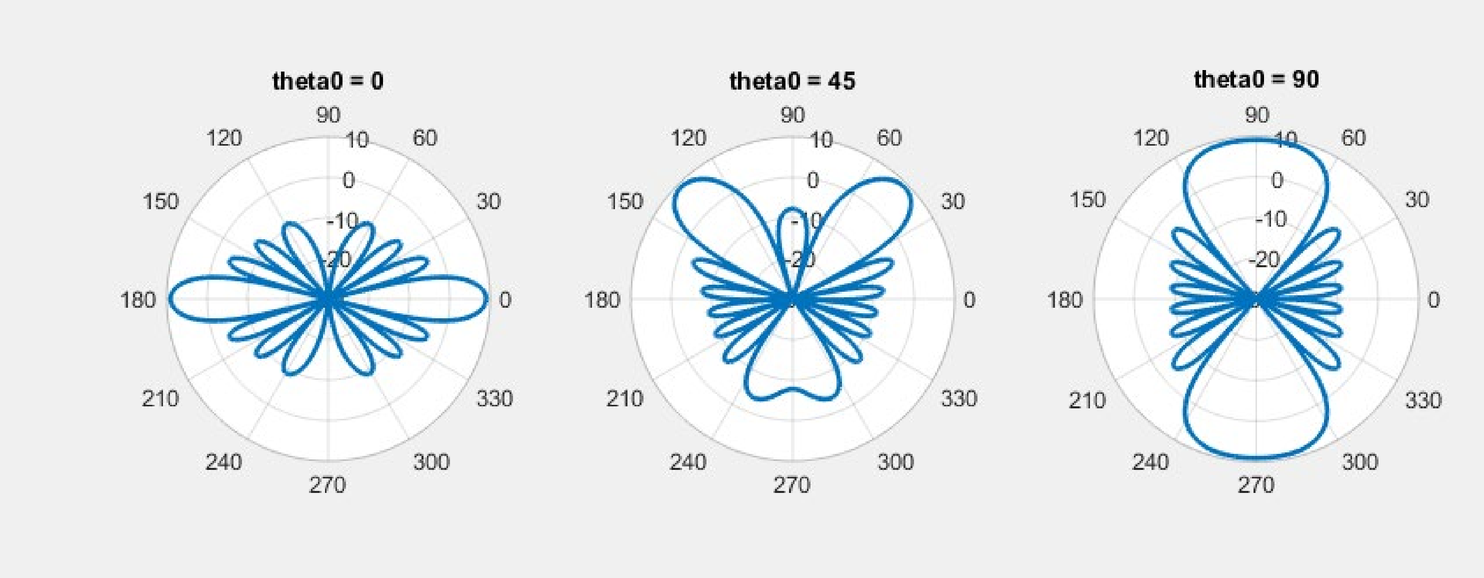



```
for iplot = 1:nplot
    % Get the SV for the beam direction.
    % Note:  You must call release method of the sv
    % before each call since it expects the same size
    % of the input
    ang0 = [azPlot(iplot); 0];
    sv.release();
    u0 = sv(fc, ang0);

    % Normalize the direction
    u0 = u0 / norm(u0);

    % Get the SV for the AoAs.  Take el=0
    npts = 1000;
    az = linspace(-180,180,npts);
    el = zeros(1,npts);
    ang = [az; el];
    sv.release();
    u = sv(fc, ang);

    % Compute the AF and plot it
    AF = 10*log10( abs(sum(conj(u0).*u, 1)).^2 );

    % Plot it
    subplot(1,nplot,iplot);
    plot(ang(1,:), AF, 'LineWidth', 2);

end
```

Polar Plot

- ❑Useful to visualize in polar plot
- ❑Note key features:
  - Direction of maximum gain
  - Sidelobes
  - Pattern repeated on reverse side

```
% Polar plot
AFmin = -30;
subplot(1,nplot,iplot);
polarplot(deg2rad(az), max(AF, AFmin),'LineWidth', 2);
rlim([AFmin, 10]);
grid on;
```



Key Statistics

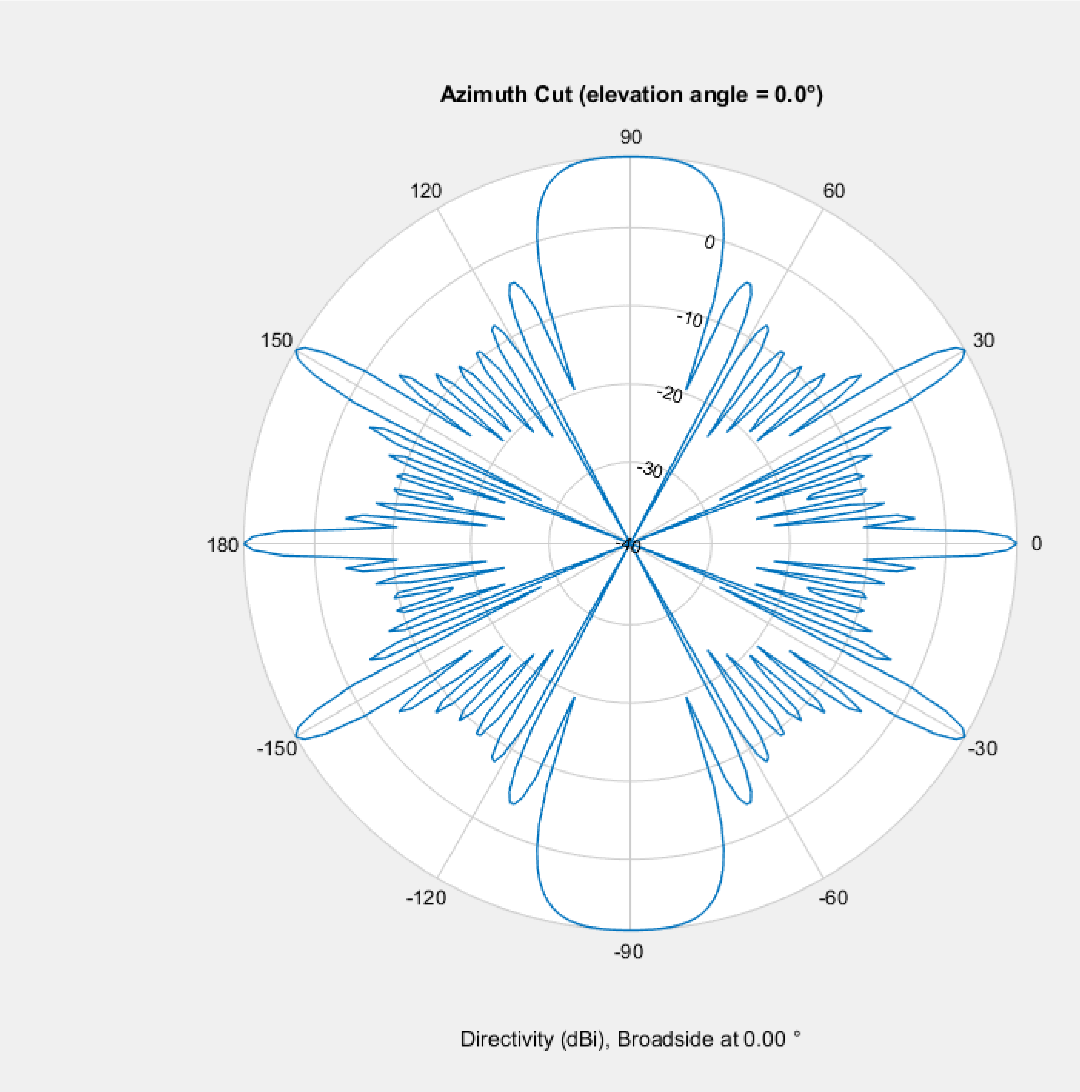
		Broadside ($\theta_0 = \pi/2$)	End-fire ($\theta_0 = 0$)
Full null beamwidth (zero to zero)	FNBW	$2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{\lambda}{N\Delta}\right)\right]$ (30°)	$2\cos^{-1}\left(1 - \frac{\lambda}{N\Delta}\right)$ (83°)
Half power beamwidth (-3dB to -3dB)	HPBW	$2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.39\lambda}{\pi N\Delta}\right)\right]$ (13°)	$2\cos^{-1}\left(1 - \frac{1.39\lambda}{\pi N\Delta}\right)$ (54°)
First sidelobe level	FSLL	$\frac{1}{N\left\lvert\sin\left(\frac{3\pi}{2N}\right)\right\rvert}$ (-13 dB)	$\frac{1}{N\left\lvert\sin\left(\frac{3\pi}{2N}\right)\right\rvert}$ (-13 dB)
	D_0	$2N\Delta/\lambda$ (9 dB)	$4N\Delta/\lambda$ (12 dB)

- From Jacobs University slides
- Values in () for: $d = \lambda/2\,,\ M = 8$

Grating Lobes

- When $d > \frac{\lambda}{2}$
- Obtain multiple peaks
- Does not direct gain in one direction

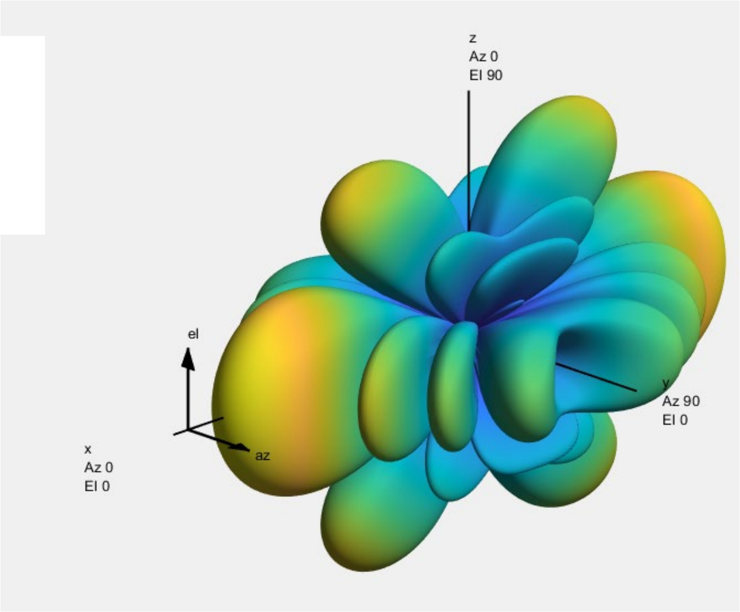
```
dsep = 2*lambda;       % element spacing
nant = 8;              % Number of elements
arr = phased.ULA(nant,dsep);

% Get the SV for the beam direction.
ang0 = [0; 0];
sv.release();
u0 = sv(fc, ang0);

arr.patternAzimuth(fc,'Weights', u0);
```



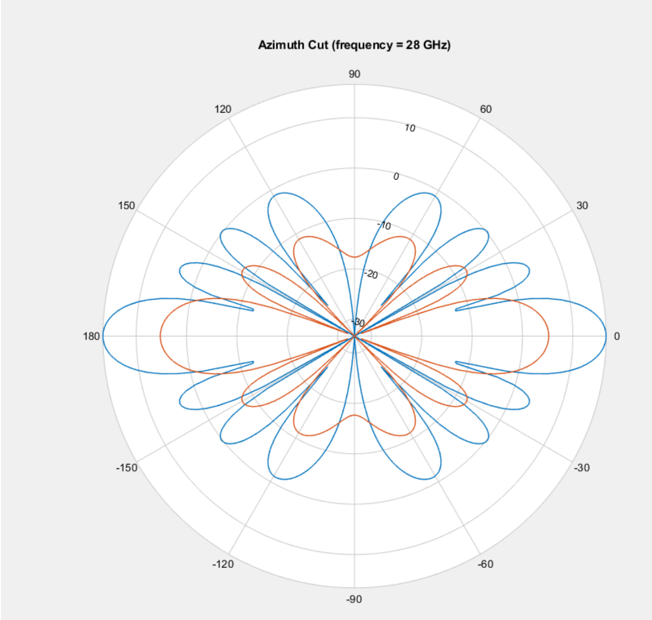
Plotting the Patterns

- MATLAB has excellent routines for 3D patterns
- Note that this plots directivity not array factor

```
sv = phased.SteeringVector('SensorArray',arr);
ang0 = [0; 0];
sv.release();
u0 = sv(fc, ang0);
u0 = u0 / norm(u0);
```




```
% We can plot the directivity in a 3D plot
arr.pattern(fc,'Weights', u0);
```




```
elPlot = [0 45];
arr.patternAzimuth(fc, elPlot, 'Weights', u0);
```

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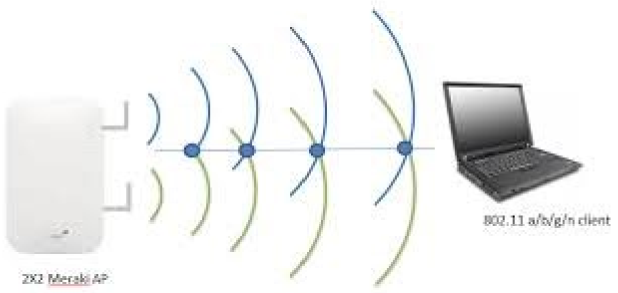
Multiple TX antennas

- MISO channel
  - Multiple input single output
  - M TX antennas, 1 RX antennas
  - Transmit vector: $\boldsymbol{x}(t) = \left(x_1(t), \ldots, x_M(t)\right)^T$
  - Scalar RX: $r(t)$



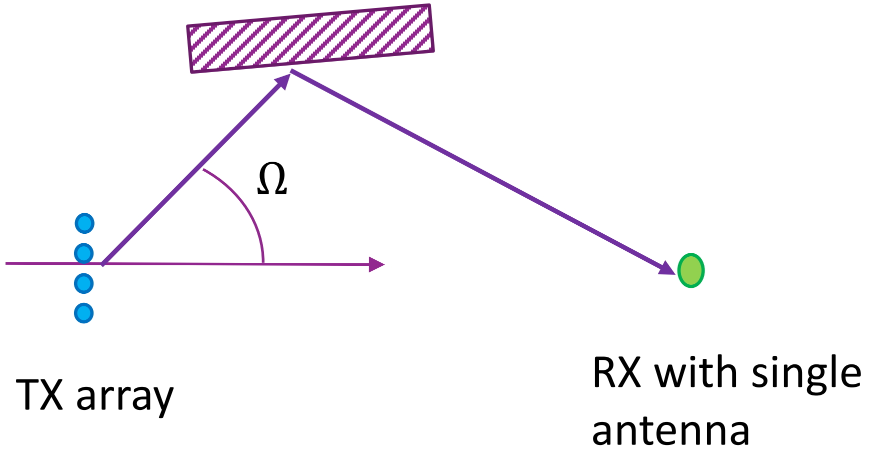
- Most of the theory is identical to the SIMO channel

Single Path Channel

- First consider single path channel
- Similar to the SIMO case, RX signal is:

$$r(t) = g_0 A(\Omega)\boldsymbol{u}^T(\Omega)\boldsymbol{x}(t-\tau)$$

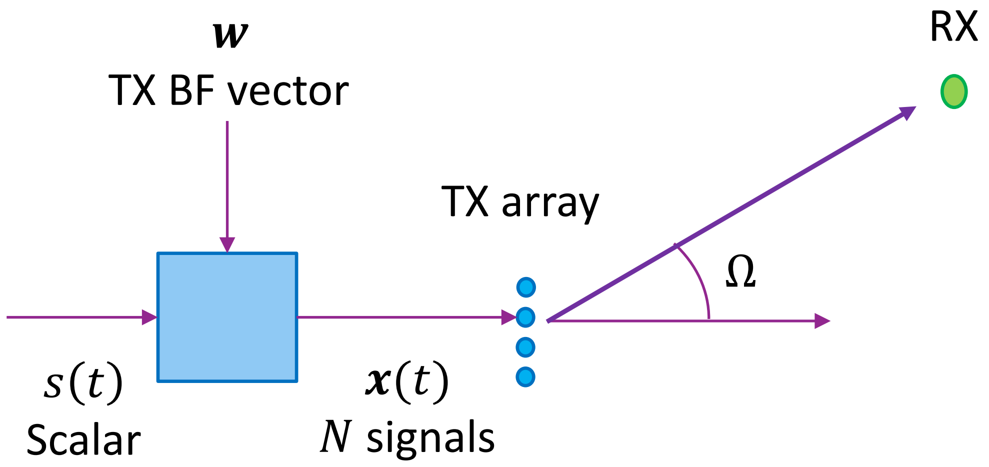
  - g_0 path gain
  - Ω = angle of departure
  - τ = path delay
  - $\boldsymbol{u}(\Omega)$ TX spatial signature
  - $A(\Omega)$: complex TX element gain



- TX and RX spatial signatures are identical

TX Beamforming

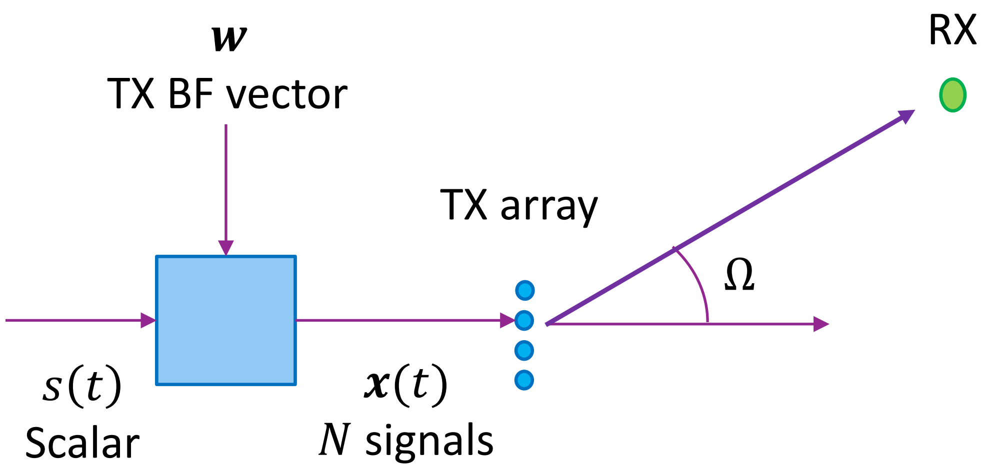
- RX signal is: $r(t) = g_0 \boldsymbol{u}^T(\Omega)\boldsymbol{x}(t-\tau) + n(t)$
- TX beamforming
  - Input scalar information signal $s(t)$
  - Create vector signal to antennas: $\boldsymbol{x}(t) = \boldsymbol{w}\, s(t)$
- Signal to antenna i is: $x_i(t) = w_i s(t)$
  - w_i is a complex weight applied to signal



- $\boldsymbol{w}$ is called the TX beamforming vector
  - Also called pre-coding

SNR with TX Beamforming

- RX signal is: $r = g_0 \boldsymbol{u}^T(\Omega)\boldsymbol{x} + n$
  - Drop dependence on time to simplify notation
- With $\boldsymbol{x} = \boldsymbol{w}s$ SISO channel is $r = g_0 \boldsymbol{u}^T(\Omega)\boldsymbol{w}s + n$
- Total transmitted energy across all N TX chains is:
  - $E_x = \sum |w_j|^2 E_s = \|\boldsymbol{w}\|^2 E_s$
  - To keep constant total energy: $\|\boldsymbol{w}\|^2 = 1$
  - Assumes no mutual coupling
- SNR is $\gamma = \frac{|g_0|^2}{N_0} E_s |\boldsymbol{u}^T(\Omega)\boldsymbol{w}|^2 = \gamma_0 |\boldsymbol{u}^T(\Omega)\boldsymbol{w}|^2$
  - $\gamma_0 = \frac{|g_0|^2}{N_0} E_s$ is the SNR for a single antenna

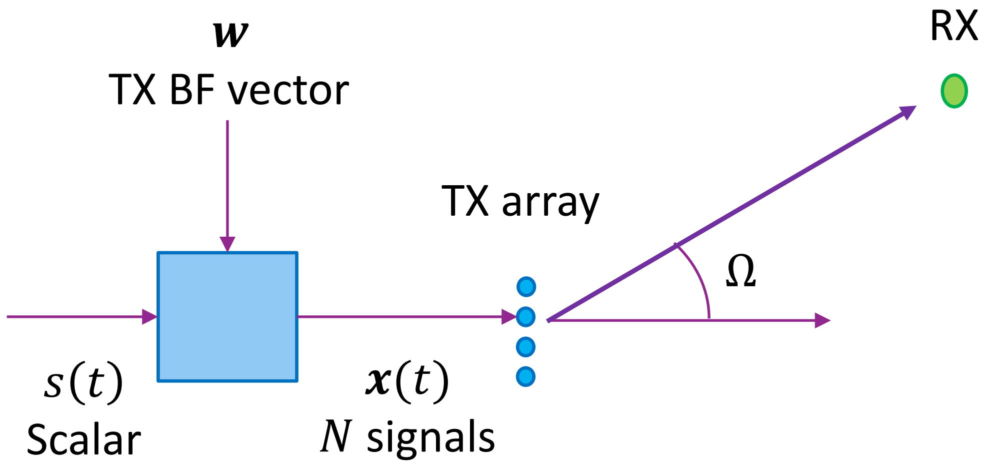


MRC TX Beamforming

- From previous slide, SNR is: $\gamma = \gamma_0 |\boldsymbol{u}^*(\Omega)\boldsymbol{w}|^2$
- To maximize SNR s.t. power constraint

$$\widehat{\boldsymbol{w}} = \arg\max |\boldsymbol{u}^T(\Omega)\boldsymbol{w}|^2 \ \text{ s.t. } \|\boldsymbol{w}\|^2 = 1$$

- MRC TX BF vector: $\widehat{\boldsymbol{w}} = \frac{1}{\sqrt{N}}\overline{\boldsymbol{u}}(\Omega)$
  - Align with the conjugate of the spatial signature
  - SNR gain $= |\boldsymbol{u}^T(\Omega)\widehat{\boldsymbol{w}}|^2 = N$
- Define and compute Array Factor similarly



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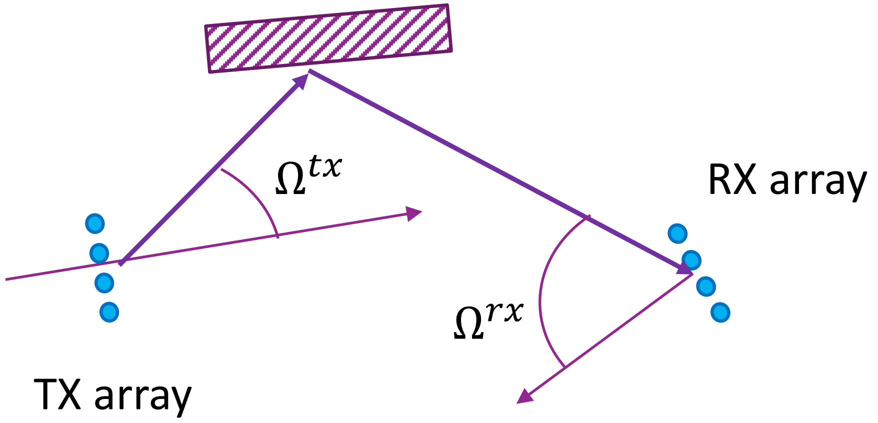
MIMO Channel with a Single Path

- Multi-input Multi-Output (MIMO) channel:
  - TX array with N_t elements
  - RX array with N_r elements



- Single path channel:

$$\boldsymbol{r}(t) = g_0 \boldsymbol{u}_{rx}(\Omega^{rx}) \boldsymbol{u}_{tx}^T(\Omega^{tx}) \boldsymbol{x}(t-\tau) = \boldsymbol{H}\boldsymbol{x}(t-\tau)$$

- MIMO channel matrix for a single path channel:

$$\boldsymbol{H} = g_0 \boldsymbol{u}_{rx}(\Omega^{rx}) \boldsymbol{u}_{tx}^T(\Omega^{tx})$$

Beamforming on a MIMO Channel

- Consider MIMO channel, $\boldsymbol{r} = \boldsymbol{Hx} + \boldsymbol{v},\ \boldsymbol{H} \in \mathbb{C}^{M\times N},\ \ \boldsymbol{v} \sim CN(\boldsymbol{0}, N_0 \boldsymbol{I})$
  - Channel on time and frequency resource
- Apply TX beamforming: $\boldsymbol{x} = \boldsymbol{w}_{tx} s$
  - Assume $\|\boldsymbol{w}_{tx}\| = 1$ so total transmit energy is $E_s = E|s|^2$
- Apply RX beamforming: $z = \boldsymbol{w}_{rx}^T \boldsymbol{r}$
  - Assume $\|\boldsymbol{w}_{rx}\| = 1$ so total received noise energy $E|\boldsymbol{w}_{rx}^T \boldsymbol{v}|^2 = \mathrm{N}_0$
- Equivalent channel: $z = \boldsymbol{w}_{rx}^T \boldsymbol{r} = Gs + d,$
  - $G = \boldsymbol{w}_{rx}^T \boldsymbol{H} \boldsymbol{w}_{tx}$ = complex beamformed channel gain
  - Noise energy is $E|\boldsymbol{w}_{rx}^T \boldsymbol{v}|^2 = \mathrm{N}_0$
- SNR with beamforming: $\gamma = \dfrac{|G|^2 E_s}{\mathrm{N}_0} = \dfrac{\left|\boldsymbol{w}_{rx}^T \boldsymbol{H} \boldsymbol{w}_{tx}\right|^2 E_s}{\mathrm{N}_0}$

Beamforming Gain with a Single Path

- From previous slide, we saw SNR on a MIMO channel is: $\gamma = \frac{|G|^2 E_s}{\mathrm{N}_0} = \frac{\left|\boldsymbol{w}_{rx}^T \boldsymbol{H} \boldsymbol{w}_{tx}\right|^2 E_s}{\mathrm{N}_0}$
- Suppose we have a single path channel: $\boldsymbol{H} = g_0 \boldsymbol{u}_{rx}(\Omega^{rx}) \boldsymbol{u}_{tx}^T(\Omega^{tx})$
- Take TX and RX conjugate beamforming vectors:
  - $\boldsymbol{w}_{rx} = \frac{\overline{\boldsymbol{u}}_{rx}(\Omega^{rx})}{\sqrt{N_r}}, \boldsymbol{w}_{tx} = \frac{\overline{\boldsymbol{u}}_{tx}(\Omega^{tx})}{\sqrt{N_t}}$
- Then SNR is $\gamma = \frac{|g_0|^2 E_s}{\mathrm{N}_0} \frac{|u_{rx}^*(\Omega^{rx}) u_{rx}(\Omega^{rx})|^2}{N_r} \frac{|u_{tx}^*(\Omega^{tx}) u_{tx}(\Omega^{tx})|^2}{N_t} = \frac{|g_0|^2 E_s}{\mathrm{N}_0} N_r N_t$
- But $\frac{|g_0|^2 E_s}{\mathrm{N}_0}$ is the SNR per antenna

Conclusion: Maximum BF gain on a single path channel is $N_r N_t$

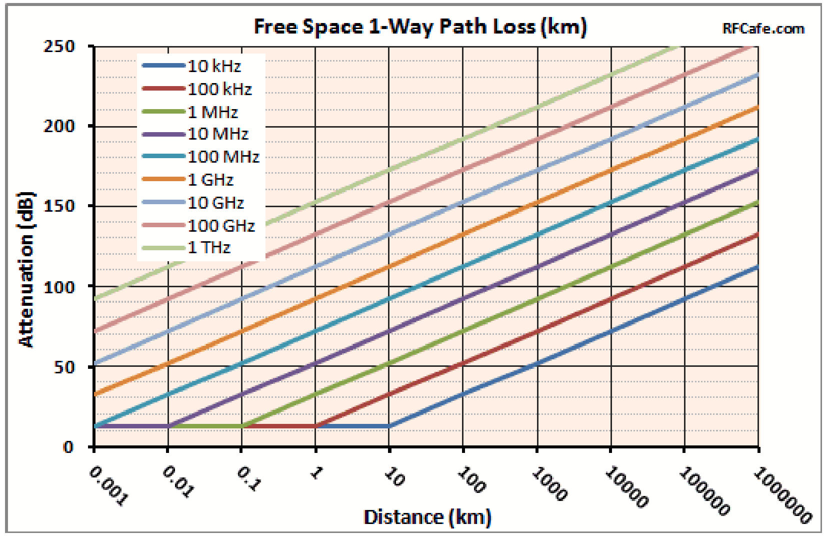
  - Again, assuming no mutual coupling

Friis' Law and MmWave

- Recall Friis' Law: $\frac{P_r}{P_t} = D_r D_t \left(\frac{\lambda}{4\pi R}\right)^2$
- Isotropic path loss decreases with λ^2
- Millimeter Wave systems: Increases f_c^2
  - Decreases λ^2 $\Rightarrow$ Increase path loss
- But, with beamforming:
  - Directivity $D_r \propto N_r$ and $D_t \propto N_t$
  - Each antenna takes area $\propto \lambda^2$
  - So, for fixed total aperture:

$$D_r \propto N_r \propto \frac{1}{\lambda^2}, D_t \propto N_t \propto \frac{1}{\lambda^2}$$

- Can compensate isotropic path loss with directivity



Friis' Law and MmWave

Condition	Directivity scaling	Path loss scaling
No beamforming	D_i constant	$PL \propto f_c^2$
Beamforming on one side (TX or RX)	$D_1 \propto f_c^2,\ D_2$ constant	PL constant
Beamforming on both sides (TX and RX)	$D_1, D_2 \propto f_c^2$	$PL \propto f_c^{-2}$

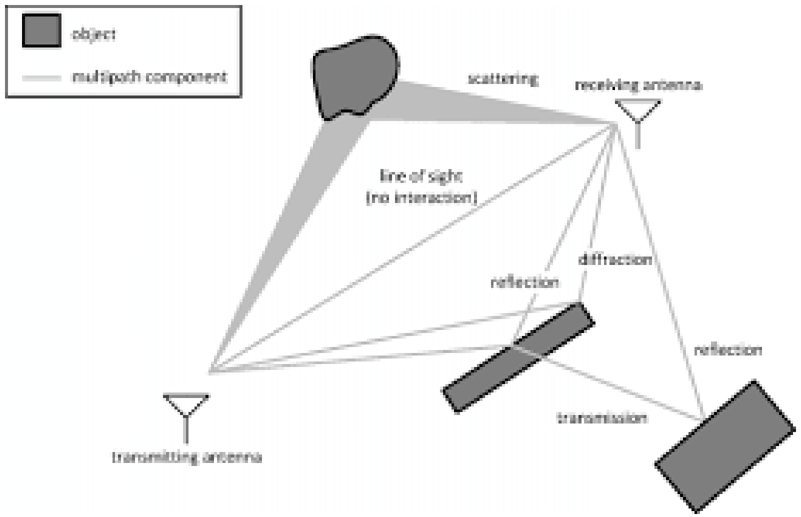
- Friis' Law: $\frac{P_r}{P_t} = D_1 D_2 \left(\frac{\lambda}{4\pi R}\right)^2$
- Conclusions: With a fixed aperture and beamforming
  - Isotropic path loss can be overcome
- But systems need very directive beams
  - Raises many other issues. E.g. Channel tracking, processing, …

Multiple Paths

- Easy to extend channel response to multiple paths
- Each path adds a term with a spatial signature
- Time-domain model

$$\boldsymbol{r}(t) = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t}\, \boldsymbol{u}_{rx}(\Omega_\ell^{rx}) \boldsymbol{u}_{tx}^T(\Omega_\ell^{tx}) x(t-\tau_\ell) + \boldsymbol{n}(t)$$

Complex gain — Doppler shift — AoA — AoD — Delay



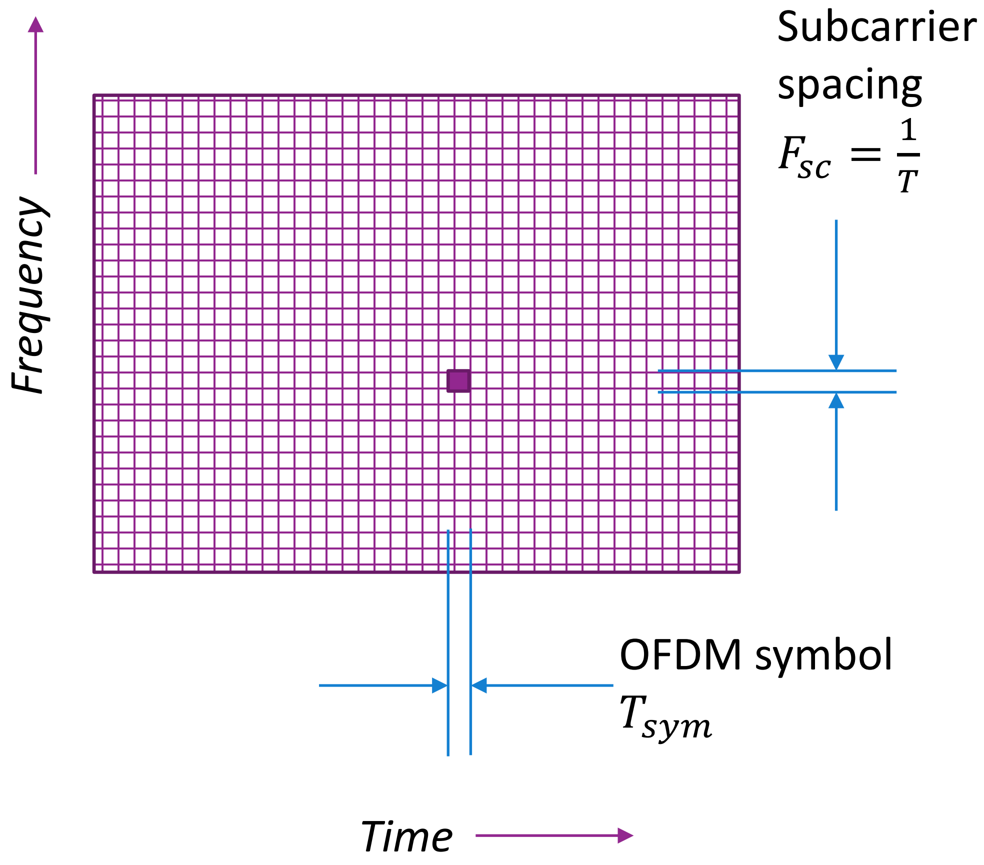
Time-Varying Frequency Response

- The channel response can also be described as a time and frequency-varying matrix

$$\boldsymbol{H}(t,f) = \sum_{\ell=1}^{L} g_\ell e^{2\pi j (f_\ell t - \tau_\ell f)} \boldsymbol{u}_{rx}(\Omega_\ell^{rx}) \boldsymbol{u}_{tx}^T(\Omega_\ell^{tx})$$

  - At time and frequency $\boldsymbol{H}(t,f) \in \mathbb{C}^{N_r \times N_t}$
  - Varies in time due to Doppler shifts f_ℓ
  - Varies in frequency due to delay spread τ_ℓ

OFDM Time-Frequency Grid



- Consider OFDM channel
  - Sub-carrier spacing F_{sc}, symbol time T_{sym}
  - Index with k = OFDM symbol index, n = subcarrier index
- Transmit array: $\boldsymbol{X}[n,k]$
  - At each k, n, we transmit a vector

$$\boldsymbol{X}[n,k] = \left[X_1[n,k], \ldots, X_N[n,k]\right]^T$$

  - N =number of TX antennas
- Receive array: $\boldsymbol{Y}[n,k]$:

$$\boldsymbol{Y}[n,k] = \left[Y_1[n,k], \ldots, Y_M[n,k]\right]^T$$

  - M = number of RX antennas
  - One M dim vector per resource element

OFDM Channel with Multiple RX Antennas

- OFDM channel acts as multiplication:
- Under normal operation (delay spread is contained in CP):

$$\boldsymbol{Y}[k,n] = \boldsymbol{H}[k,n] \;\; \boldsymbol{X}[k,n]$$

RX symbol vector — Channel matrix — TX symbol vector

- OFDM channel gains can be computed from the multi-path components

$$\boldsymbol{H}[k,n] = \sum_{\ell=1}^{L} g_\ell e^{2\pi j\,(T_{sym} k f_\ell - F_{sc} n \tau_\ell)}\, \boldsymbol{u}_{rx}(\Omega_\ell^{rx})\boldsymbol{u}_{tx}^*(\Omega_\ell^{tx})$$

  - T = OFDM symbol time, S = sub-carrier spacing
  - For each path: f_ℓ =Doppler shift, τ_ℓ =Delay, g_ℓ =complex gain

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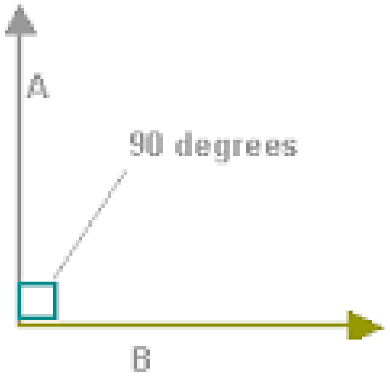
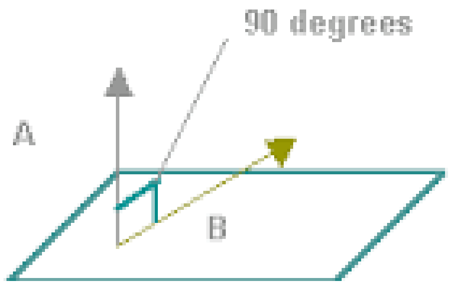
Orthogonal Vectors

- Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (real or complex)
- Vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^N$ are orthogonal if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^* \boldsymbol{y} = 0.$
  - Write $\boldsymbol{x} \perp \boldsymbol{y}$
  - Visually, $\boldsymbol{x} \perp \boldsymbol{y}$ if they are at 90 degrees
- A set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_K \in \mathbb{F}^N$ are orthonormal
  - $\boldsymbol{v}_i \perp \boldsymbol{v}_j$ when $i \neq j$
  - $||\boldsymbol{v}_i|| = 1$ for all i
  - Vectors are pairwise orthogonal and unit norm
- Orthonormal basis: An orthonormal set $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N \in \mathbb{F}^N$
  - Any vector can be written $x = \sum \alpha_n \boldsymbol{v}_n, \; \alpha_n = \boldsymbol{v}_n^* \boldsymbol{x}$
  - α_n are the coefficients of $\boldsymbol{x}$ in the basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$





Orthogonal and Unitary Matrices

- A matrix $U \in \mathbb{C}^{N \times N}$ is unitary if $U^* U = U U^* = I$
- A matrix $U \in \mathbb{R}^{N \times N}$ is orthogonal if $U^T U = U U^T = I$
  - Orthogonal is just the real-valued version of unitary
- Key properties:
  - U is orthogonal if and only if columns are orthonormal
  - U is orthogonal if and only if rows are orthonormal
  - Taking an inverse is easy $U^{-1} = U^*$

Examples of Orthogonal Matrices

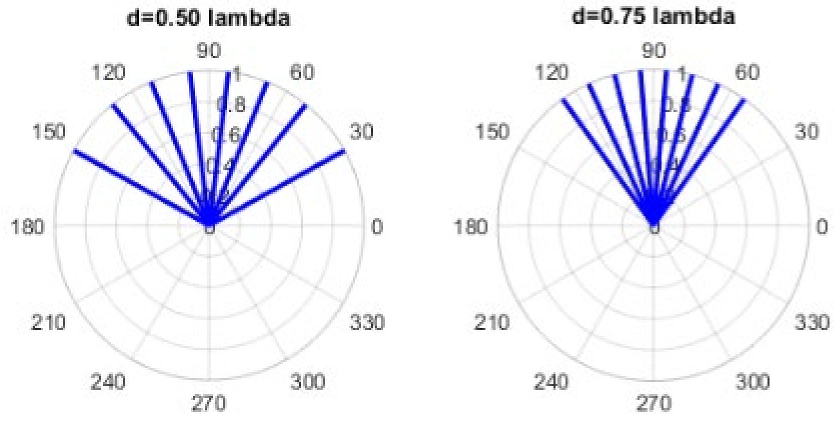
- 2D rotation matrix by θ: $\boldsymbol{V} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
  - Can verify that $\boldsymbol{V}^*\boldsymbol{V} = I$
  - 3D rotation matrices are also orthogonal
- Example with 3 vectors:
  - Let $\boldsymbol{v}_1 = \frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_3 = \frac{1}{\sqrt{66}}\begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$
  - Can verify that $\boldsymbol{v}_i^*\boldsymbol{v}_j = \delta_{ij}$
  - Hence the matrix: $\boldsymbol{V} = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$

Beamspace Matrices

- Consider a ULA with normalized steering vector:

$$\boldsymbol{u}(\phi) = \frac{1}{\sqrt{N}}\left[1, e^{j\beta\cos\phi}, \dots, e^{j(N-1)\beta\cos\phi}\right]^T, \qquad \beta = \frac{2\pi d}{\lambda}$$

- Take N angles: $\beta\cos\phi_n = 2\pi\left(\frac{n}{N} - \frac{1}{2} + \frac{1}{N}\right),\ \ n = 0,1,\dots,N-1$
  - This is possible if $d \geq \frac{\lambda}{2}$
- The vectors $\boldsymbol{u}(\phi_n), n = 0,1,\dots,N-1$ are orthonormal
- These are called the beamspace vectors
  - An orthonormal basis for the spatial domain



Symmetric and Hermitian Matrices

- Definition:
  - A matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ is symmetric if $\boldsymbol{A} = \boldsymbol{A}^T$
  - A matrix $\boldsymbol{A} \in \mathbb{C}^{N\times N}$ is Hermitian if $\boldsymbol{A} = \boldsymbol{A}^*$
- Symmetric is the real version of Hermitian
- For any $\boldsymbol{A}$ symmetric / Hermitian:
  - There are an orthonormal set of eigenvectors $\boldsymbol{v}_1, . . , \boldsymbol{v}_N$ with eigenvalues $\lambda_1, \dots, \lambda_N$
  - All eigenvalues are real (not complex)
- Let $\boldsymbol{V} = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_N] \in \mathbb{F}^{N\times N}$ = Matrix with the eigenvectors as the columns
  - Then $\boldsymbol{V} = \boldsymbol{V}^*$ is orthogonal / unitary
  - Hence $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^*, \boldsymbol{D} = diag(\lambda_1, \dots, \lambda_N)$ diagonalizable with unitary

Sample Problem

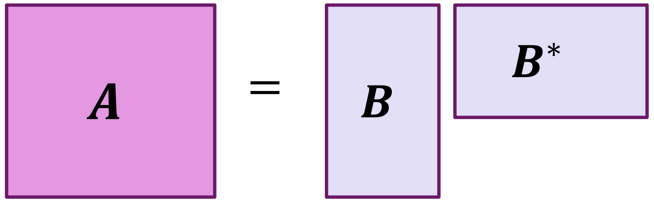
- Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal bases of eigenvectors and their eigenvalues
- Solution: Eigenvalues:
  - $\det(\lambda I - A) = det \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2 - 4 = 0$
  - $\lambda = 1 \pm 2 = -1,3$
- For $\lambda = -1$, $(\lambda I - A)v = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -v_2$
  - Take $v = \frac{1}{\sqrt{2}}[1, -1]^T$
- For $\lambda = 3$, $(\lambda I - A)v = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2$
  - Take $v = \frac{1}{\sqrt{2}}[1,1]^T$

Positive Definite Matrices

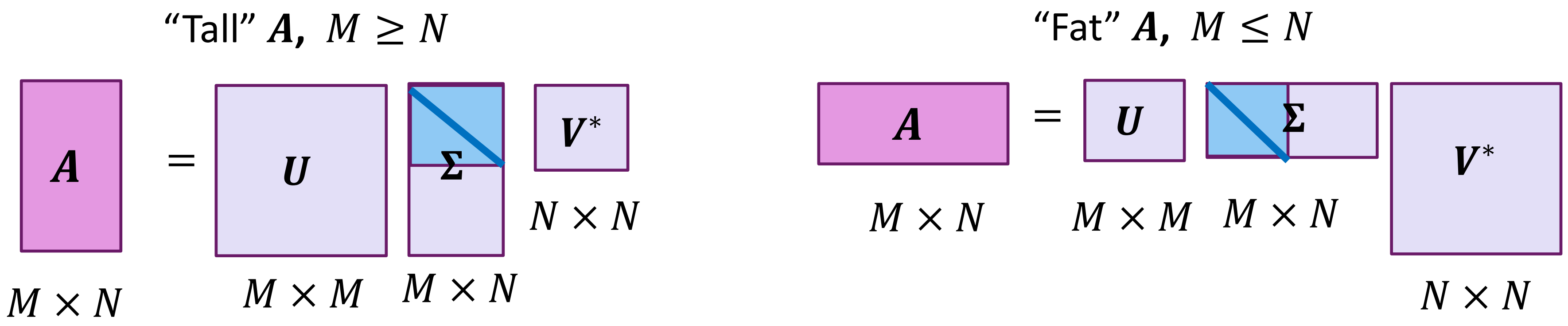
- Let $\boldsymbol{A} = \boldsymbol{A}^* \in \mathbb{F}^{N \times N}$ be symmetric / Hermitian with eigenvalues $\lambda_1, \dots, \lambda_N$
  - Recall that the eigenvalues are real
- Definition:
  - A is positive semi-definite if $\lambda_i \geq 0$ for all i
  - A is positive definite if $\lambda_i > 0$ for all i
- Notation: $\boldsymbol{A} > 0$ for positive definite and $\boldsymbol{A} \geq 0$ when $\boldsymbol{A}$ is positive semi-definite
- Key property: If $\boldsymbol{A} = \boldsymbol{A}^*$ then:
  - $\boldsymbol{A} \geq 0$ if and only if $\boldsymbol{x}^* \boldsymbol{A} \boldsymbol{x} \geq 0$ for all $\boldsymbol{x}$
  - $\boldsymbol{A} > 0$ if and only if $\boldsymbol{x}^* \boldsymbol{A} \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq 0$

Matrix Square Roots

- **Theorem**: Let $A \in \mathbb{F}^{N\times N}$. Then $A \geq 0$ if and only if $A = BB^*$ for some $B \in \mathbb{F}^{N\times M}$
  - Note: The dimension M can be anything $(M \geq N$ or $M < N)$
- Proof:
  - $(\Rightarrow)$ Suppose $A \geq 0$. Then $A = UDU^*, D = diag(\lambda_1, \ldots, \lambda_N)$
  - Write $B = UD^{1/2}U^*$. $D = diag(\lambda_1^{1/2}, \ldots, \lambda_N^{1/2})$
  - Then: $BB^* = B^2 = UD^{1/2}U^* \, UD^{1/2}U^* = UDU^* = A$
  - Since $A = B^2$ and $B \geq 0$, B is called the matrix square root. Write $B = A^{1/2}$
  - $(\Leftarrow)$ Suppose $A = BB^*$.
  - Then for any x, $x^*Ax = x^*BB^*x = \|B^*x\|^2 \geq 0$



Singular Value Decomposition



- Given matrix $\boldsymbol{A} \in \mathbb{F}^{M\times N}$, an SVD is a factorization of the form, $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$ where
  - $\boldsymbol{U} \in \mathbb{F}^{M\times M}$, $\boldsymbol{U}^*\boldsymbol{U} = \boldsymbol{I}_M$, a unitary matrix
  - $\boldsymbol{V} \in \mathbb{F}^{N\times N}$, $\boldsymbol{V}^*\boldsymbol{V} = \boldsymbol{I}_N$, a unitary matrix
  - If $M \geq N$, $\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{diag}(\sigma_1, \ldots, \sigma_N) \\ \boldsymbol{0}_{(M-N)\times N} \end{bmatrix}$. If $N \geq M$, $\boldsymbol{\Sigma} = [\mathrm{diag}(\sigma_1, \ldots, \sigma_M) \quad \boldsymbol{0}_{N\times(M-N)}]$
- Values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_L \geq 0$, $L = min(M, N)$. Called the singular values
- All matrices have an SVD

Example

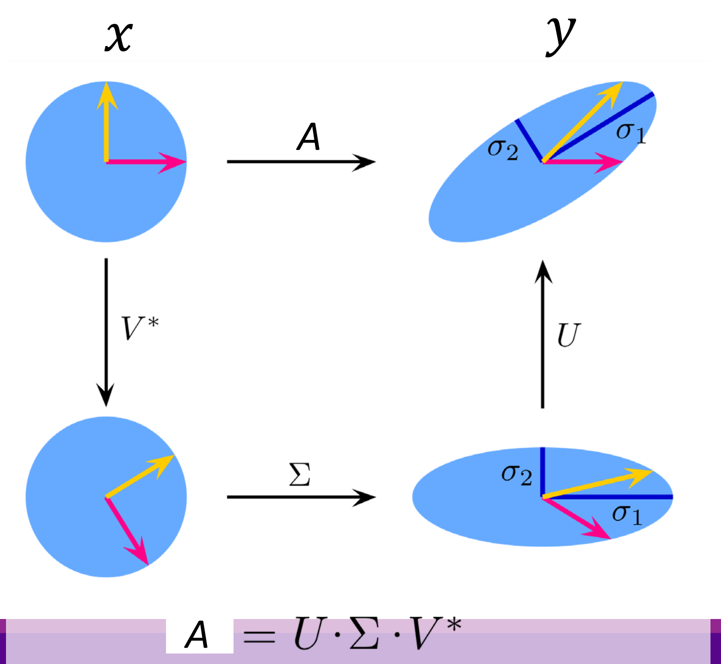
- Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}$
- Then can check that $A = U\Sigma V^*$

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

  - Also verify that $UU^* = I_5$ and $VV^* = I_5$
  - This can be found by (cleverly) permute the rows of A
  - But, in general, use a computer to compute SVD

Geometric Interpretation

- Let $A = U\Sigma V^*$ and $y = Ax$
- Consider a transformed space
  - $\boldsymbol{w} = \boldsymbol{V}^*\boldsymbol{x}$ so $\boldsymbol{w} = [w_1, \dots, w_N]$ are the coefficients of the input in the basis $V = [v_1, \dots, v_N]$
  - $\boldsymbol{z} = \boldsymbol{U}^*\boldsymbol{y}$ so $\boldsymbol{z} = [z_1, \dots, z_M]$ are the coefficients in the basis $U = [u_1, \dots, u_M]$
- Then: $\boldsymbol{z} = \boldsymbol{\Sigma}\boldsymbol{w}$ so $z_i = \sigma_i w_i$
- Each input direction $\boldsymbol{v}_i$ is mapped to $\sigma_i \boldsymbol{u}_i$
- Consequence:
  - SVD finds orthonormal bases U, V such that matrix A is a linear scaling in each basis vector

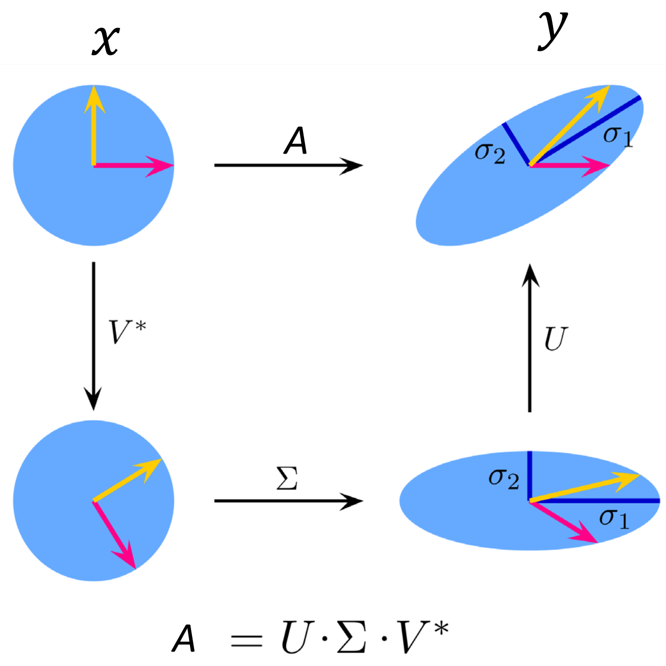


SVD and Rank

- **Theorem**: Suppose $\boldsymbol{A} = \boldsymbol{U\Sigma V}^* \in \mathbb{F}^{M\times N}$, then

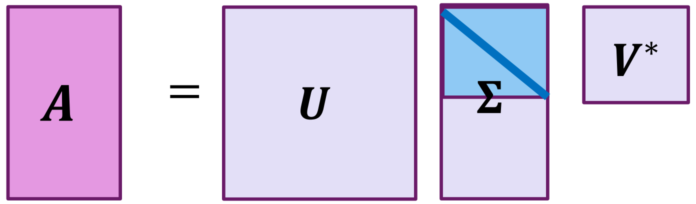
$$rank(\boldsymbol{A}) = |\{\sigma_\ell > 0\}| = \text{num of positive singular values}$$

- Ex: Suppose $\boldsymbol{A} \in \mathbb{C}^{5\times 3}$ with $\sigma = \{10,2,0\}$
  - Then: $rank(\boldsymbol{A}) = 2$
- Proof:
  - For any $\boldsymbol{x}$, the output is $\boldsymbol{y} = \boldsymbol{Ax} = \boldsymbol{U\Sigma V}^*\boldsymbol{x}$
  - Define $\boldsymbol{z} = \boldsymbol{U}^*\boldsymbol{y}$ and $\boldsymbol{w} = \boldsymbol{V}^*\boldsymbol{x}$
  - Then $z_\ell = \sigma_\ell w_\ell$
  - If $r = |\{\sigma_\ell > 0\}|$, then $\sigma_\ell > 0$ for $\ell = 1, \ldots, r$
  - Hence, by varying w_ℓ, we can span a space of dimension r



Sum of Rank One Form

- Suppose $\boldsymbol{A} = \boldsymbol{U\Sigma V}^* \in \mathbb{F}^{M\times N}$ with $r = rank(\boldsymbol{A})$



- Then:

$$\boldsymbol{A} = \sum_{\ell=1}^{r} \sigma_\ell \boldsymbol{u}_\ell \boldsymbol{v}_\ell^*$$

  - A sum of rank one terms $\boldsymbol{u}_\ell \boldsymbol{v}_\ell^*$
- The vectors $\boldsymbol{u}_\ell, \ell = 1, \ldots, r$ are an orthonormal basis for $Range(\boldsymbol{A})$
- The vectors $\boldsymbol{v}_\ell, \ell = 1, \ldots, r$ are an orthonormal basis for $Range(\boldsymbol{A}^*)$

Outline

- Antenna Arrays and the Spatial Signature
- Receive Beamforming and SNR Gain with a Single Path
- Array Factor
- Transmit Beamforming with a Single Path
- Multipath and MIMO Channels
- Linear Algebra and SVD Review
- Beamforming Gains in Multipath Channels
- Adding Element Gains and Normalizing Spatial Signatures for Mutual Coupling

SVD of the Channel Matrix

- Consider a MIMO channel matrix: $\boldsymbol{H} = \sum_{\ell=1}^{L} \sqrt{E_\ell} e^{\theta_\ell} \boldsymbol{u}_{rx}(\Omega_\ell^{rx}) \boldsymbol{u}_{tx}^T(\Omega_\ell^{tx})$
  - $E_\ell =$ RX energy per antenna on path ℓ
  - $\theta_\ell =$ phase that varies with frequency and time
- We can write this as: $\boldsymbol{H} = \sum_{\ell=1}^{L} \sigma_\ell \, \widehat{\boldsymbol{u}}_\ell \widehat{\boldsymbol{v}}_\ell^*$ where
  - $\widehat{\boldsymbol{u}}_\ell = \frac{1}{\sqrt{N_{rx}}} e^{\theta_\ell} \boldsymbol{u}_{rx}(\Omega_\ell^{rx})$ and $\widehat{\boldsymbol{v}}_\ell = \frac{1}{\sqrt{N_{tx}}} \overline{\boldsymbol{u}}_{tx}(\Omega_\ell^{tx})$= normalized steering vectors
  - $\sigma_\ell = \sqrt{E_\ell N_{rx} N_{tx}}$
- Interpretation:
  - L =number of paths = rank of H
  - If the signatures $\widehat{\boldsymbol{u}}_\ell$ and $\widehat{\boldsymbol{v}}_\ell$ are orthogonal then they are the left and right singular vectors
  - In this case, singular values squared $\sigma_\ell^2 = E_\ell N_{rx} N_{tx} =$ RX energy × beamforming gain

Beamforming on a MIMO Channel

- Consider MIMO channel, $\boldsymbol{r} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v},\ \boldsymbol{H} \in \mathbb{C}^{M \times N},\ \ \boldsymbol{v} \sim CN(\boldsymbol{0}, N_0 \boldsymbol{I})$
  - Channel on time and frequency resource
- Apply TX beamforming: $\boldsymbol{x} = \boldsymbol{w}_{tx} s$
  - Assume $\|\boldsymbol{w}_{tx}\| = 1$ so total transmit energy is $E_s = E|s|^2$
- Apply RX beamforming: $z = \boldsymbol{w}_{rx}^T \boldsymbol{r}$
  - Assume $\|\boldsymbol{w}_{rx}\| = 1$ so total received noise energy $E|\boldsymbol{w}_{rx}^T \boldsymbol{v}|^2 = \mathrm{N}_0$
- Equivalent channel: $z = \boldsymbol{w}_{rx}^T \boldsymbol{r} = Gs + d,$
  - $G = \boldsymbol{w}_{rx}^T \boldsymbol{H} \boldsymbol{w}_{tx}$ = complex beamformed channel gain
  - Noise energy is $E|\boldsymbol{w}_{rx}^T \boldsymbol{v}|^2 = \mathrm{N}_0$
- SNR with beamforming: $\gamma = \frac{|G|^2 E_s}{\mathrm{N}_0} = \frac{\left|\boldsymbol{w}_{rx}^T \boldsymbol{H} \boldsymbol{w}_{tx}\right|^2 E_s}{\mathrm{N}_0}$

Maximizing the SNR

- From previous slide, MIMO channel with beamforming is $z = Gs + d$,
  - Gain: $G = \boldsymbol{w}_{rx}^T \boldsymbol{H} \boldsymbol{w}_{tx}$
  - Noise energy $E|d|^2 = N_0$
  - SNR: $\gamma = \frac{|G|^2 E_s}{\mathrm{N}_0} = \frac{|\boldsymbol{w}_{rx}^* \boldsymbol{H} \boldsymbol{w}_{tx}|^2 E_s}{\mathrm{N}_0}$
- Want to select the beamforming vectors to maximize the SNR:

$$\max_{\boldsymbol{w}_{rx}, \boldsymbol{w}_{tx}} |\boldsymbol{w}_{rx}^T \boldsymbol{H} \boldsymbol{w}_{tx}|^2 \quad \mathrm{s.t.}\ \|\boldsymbol{w}_{tx}\| = \|\boldsymbol{w}_{rx}\| = 1$$

- Theorem: Let $\boldsymbol{H} = \boldsymbol{U\Sigma V}^*$ be the SVD. Then, then the optimal vectors are
  - $\boldsymbol{w}_{rx} = \overline{\boldsymbol{u}}_1$ = conjugate of the left singular vector for maximal singular value
  - $\boldsymbol{w}_{tx} = \overline{\boldsymbol{v}}_1$ = conjugate of the right singular vector for maximal singular value

  Also, the max value is σ_1^2 = maximum singular value squared

CSI Requirements

- Optimal BF vectors are maximal singular vectors of channel matrix $\boldsymbol{H}$
- Problem: TX and RX must know $\boldsymbol{H}$ exactly
  - Channel state information (CSI) must be available at TX and RX
  - In general, $\boldsymbol{H}$ varies with time and frequency
  - Hence channel needs to be tracked!
- Next lecture we will discuss:
  - How to track channel in practical systems
  - Methods to approximate beamforming if exact tracking is not possible

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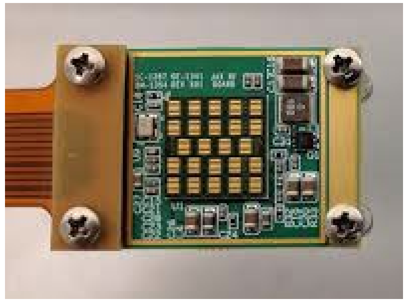
Modeling the Element Pattern

- Above analysis assumes each element is omni-directional
- However, in most systems, each antenna element may also have gain.
- In this section, we describe two methods to account for element gain
- Method 1. Pattern multiplication without normalization
    - Provides a simple approximation of the channel response
    - But neglects mutual coupling
- Method 2. Pattern multiplication with normalization
    - More accurate
    - Partially accounts for mutual coupling



SiBeam 60 GHz array

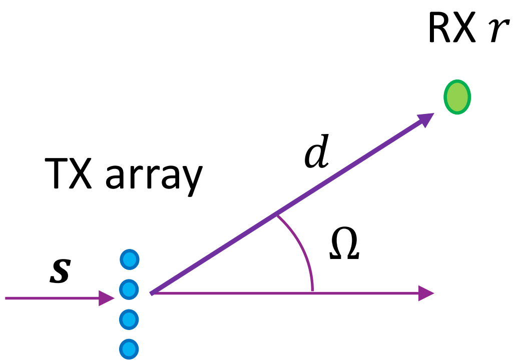
12 TX and 12 RX elements.

Uncoupled Array Assumption

- Consider a TX array with N elements in free space
  - Analysis for RX is similar
- In isolation, we know each TX signal s_n will produce an RX signal

$$r = g_0 s_n u_n(\Omega) A_E(\Omega)$$

  - g_0 = free space path gain from a reference location
  - $u_n(\Omega)$ = phase shift due to the element location relative to reference
  - $A_E(\Omega)$ = complex element gain (assumed common for all elements)



- Uncoupled array assumption:

The response from the N antennas together is given by super-position

$$r = \sum_{n=1}^{N} g_0 s_n u_n(\Omega) A_E(\Omega) = g_0 A_E(\Omega) \boldsymbol{s}^T \boldsymbol{u}(\Omega)$$

  - This is the assumption we have made implicitly up to now

Pattern Multiplication

- Previous slide shows that ignoring mutual coupling, the TX channel response is:

$$\boldsymbol{h} \approx g_0 \boldsymbol{v}_0^T(\Omega), \qquad \boldsymbol{v}_0(\Omega) = A_E(\Omega)\boldsymbol{u}(\Omega)$$

- We call $\boldsymbol{v}(\Omega)$ the pattern multiplication signature or un-normalized spatial signature
  - Multiplication of the array spatial signature with the element pattern
- Key properties:
  - TX channel is $\boldsymbol{h} = g_0 \boldsymbol{v}(\Omega)$
  - Optimal BF vector $\boldsymbol{w}(\Omega) = \frac{1}{\|\boldsymbol{v}_0(\Omega)\|}\overline{\boldsymbol{v}}_0(\Omega) = \frac{1}{\sqrt{M}}\overline{\boldsymbol{u}}(\Omega)$
  - Optimal BF gain $|\boldsymbol{w}(\Omega)^T \boldsymbol{v}_0(\Omega)|^2 = |A_E(\Omega)|^2 M = \text{peak element gain} \times \text{peak array gain}$
  - Array factor is $AF(\Omega, \Omega_0) = |\boldsymbol{w}(\Omega_0)^T \boldsymbol{v}_0(\Omega)|^2 = \frac{1}{M}|A_E(\Omega)|^2 |\boldsymbol{u}^*(\Omega_0)\boldsymbol{u}(\Omega)|^2$

Impedance and Resistance Matrices

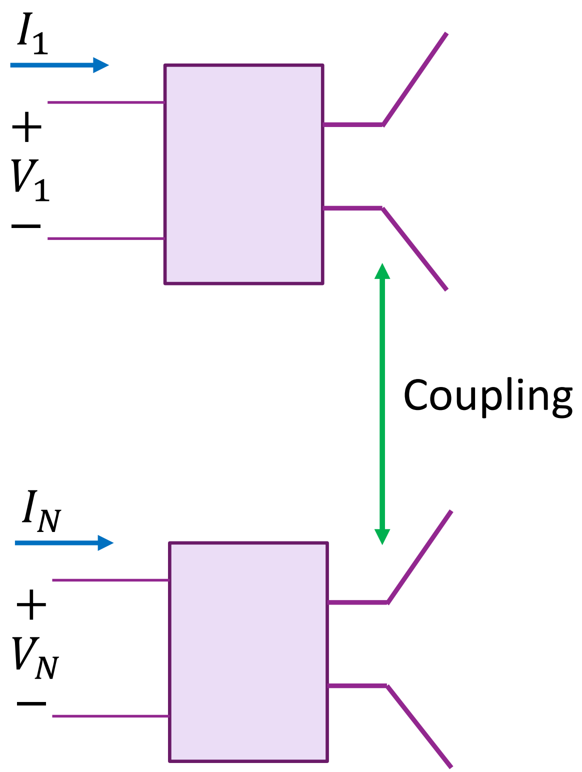
- To model mutual coupling, we need some simple network theory
- The input to an array can be modeled as an N port network
  - Each "port" has an input current I_n and voltage V_n
  - Physically, the port would be the antenna feed
  - The currents and voltages are represented in complex baseband
- Any N port network is characterized by an $N \times N$ impedance matrix $\boldsymbol{Z}$

$$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{I}$$

  - $\boldsymbol{I}$ and $\boldsymbol{V}$ are the vector of currents and voltages
  - The impedance matrix accounts for coupling between ports
- The real power consumed in the network is

$$P = \frac{1}{2} Real(\boldsymbol{I}^*\boldsymbol{V}) = \frac{1}{2} Real(\boldsymbol{I}^*\boldsymbol{Z}\boldsymbol{I}) = \frac{1}{2}\boldsymbol{I}^*\boldsymbol{R}\boldsymbol{I}$$

  - $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{Z} + \boldsymbol{Z}^*)$ =Hermitian part of Z. Called the resistance matrix



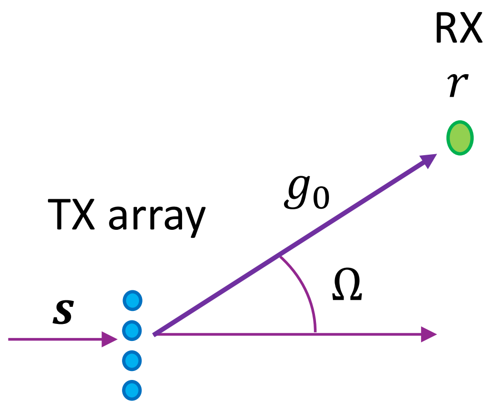
Normalized Steering Vector

- To account for coupling between antennas, define the normalized spatial signature

$$\boldsymbol{v}(\Omega) = \boldsymbol{Q}^{-1/2} A_E(\Omega)\boldsymbol{u}(\Omega), \qquad \boldsymbol{Q} = \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} |A_E(\Omega)|^2\, \boldsymbol{u}(\Omega)\boldsymbol{u}^*(\Omega) \cos\theta \, d\theta d\phi$$

  - $\boldsymbol{v}(\Omega)$ is a scaled version of the spatial signature with pattern multiplication $\boldsymbol{v}_0(\Omega)$
  - $\boldsymbol{Q}$ is called the normalization matrix, $\boldsymbol{Q}^{-1/2}$ = inverse of the matrix square root

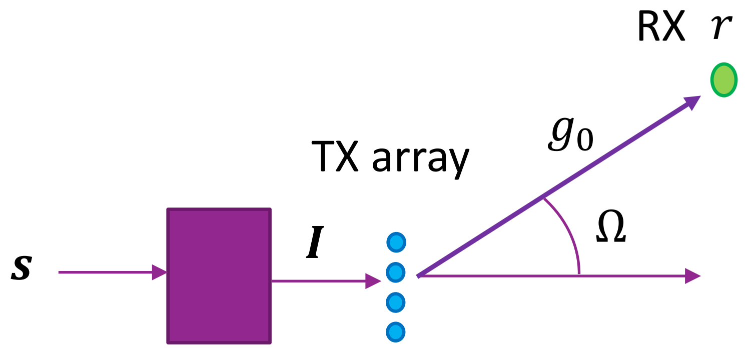
- Theorem: The TX channel in free space is $\boldsymbol{h} = g_0 \boldsymbol{v}^T(\Omega)$
  - Recall, g_0 is the free space channel from the reference point in the array
  - Proved below using network theory



- Conclusion: $\boldsymbol{v}(\Omega)$ represents the array response
  - Properly accounts for coupling between elements

Normalized Channel Response

- Theorem: There exists a constant $C > 0$ such that if $\boldsymbol{s} = \sqrt{C}\overline{\boldsymbol{Q}}^{1/2}\mathbf{I}$:
  - The total transmitted power is $||\boldsymbol{s}||^2$
  - The received signal at a point in free space is $r = g_0\boldsymbol{v}^T(\Omega)\boldsymbol{s}$ where g_0 is the free space SISO channel
  - Received power is $|r|^2 = |g_0|^2|\boldsymbol{v}^T(\Omega)\boldsymbol{s}|^2$
- Proof: Will be done in several slides below



- Conclusion: $\boldsymbol{v}(\Omega)$ represents the effective array response
  - Properly accounts for coupling between elements
  - Based on a transformation of the signals to array

Numerical Procedure for Normalization

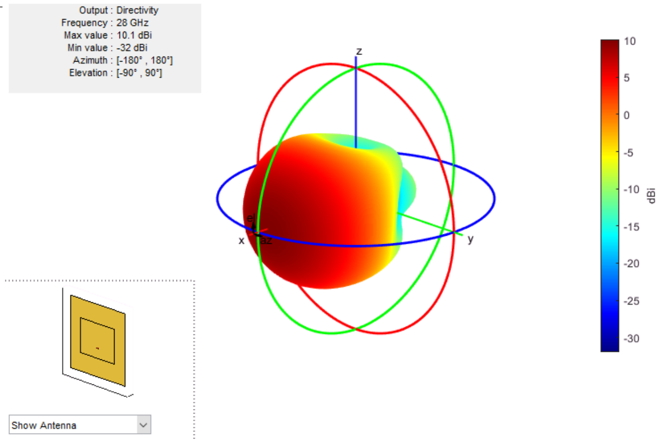
- Get angles $\Omega_k = (\theta_k, \phi_k), k = 1, \ldots, K$ uniformly in $\theta_k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\phi_k \in [-\pi, \pi]$
- Get steering vectors $\boldsymbol{u}(\Omega_k)$ and element gain $A_E(\Omega_k)$ at each angle
- Compute normalization matrix:

$$\boldsymbol{Q} = \frac{1}{cK}\sum_{k=1}^{K} \cos\theta_k \; |A_E(\Omega_k)|^2 \boldsymbol{u}(\Omega_k)\boldsymbol{u}^*(\Omega_k), \qquad c = \frac{1}{K}\sum_{k=1}^{K} \cos\theta_k$$

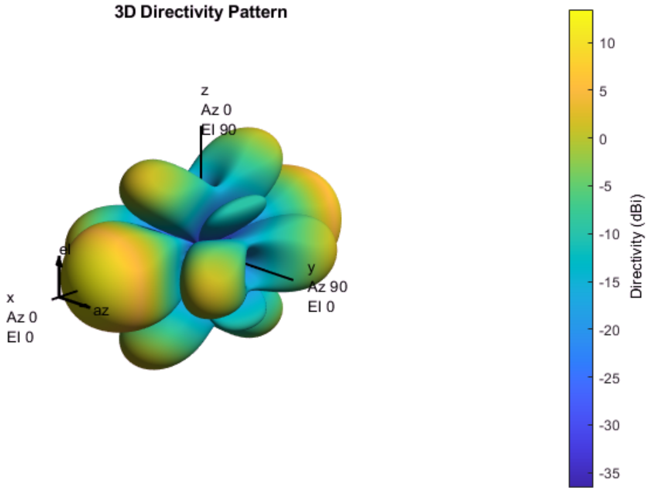
  - Scale factor c used to normalize the summation
- The normalized steering vector at any new angle Ω is $\boldsymbol{v}(\Omega) = A_E(\Omega)\boldsymbol{Q}^{-1/2}\boldsymbol{u}(\Omega)$
- The complex gain with beamforming vector $\boldsymbol{w}$ is $\boldsymbol{w}^T\boldsymbol{v}(\Omega)$
  - Power gain $G = |\boldsymbol{w}^T\boldsymbol{v}(\Omega)|^2$

Array Element Example

- Element:
  - Patch Microstrip
  - Max gain 10 dBi gain



- Array: 4x4 URA
  - Max gain = $10\log_{10} 16 = 12$ dBi
  - Has directivity in back and front



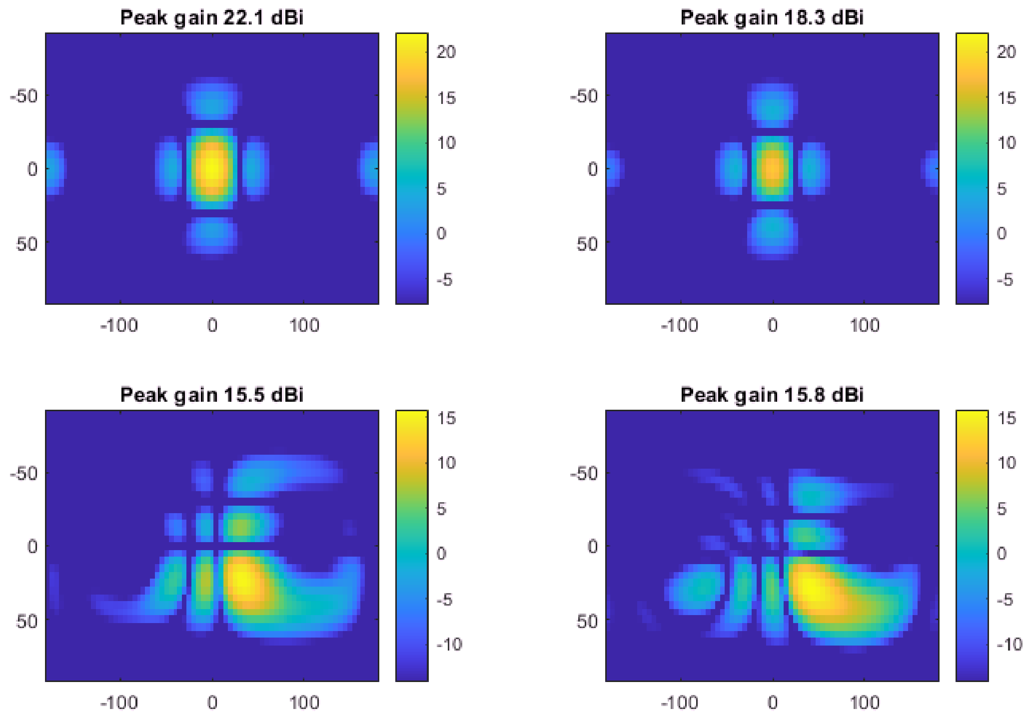
Array Factor Examples

- ❑ For each target angle:
  - Find optimal BF vector
  - Compute resulting array factor
- ❑ Array factor computed for
  - No normalization (approximate)
  - Normalization
- ❑ We see approximation is close
  - But overestimates peak gain

No normalization | Normalization

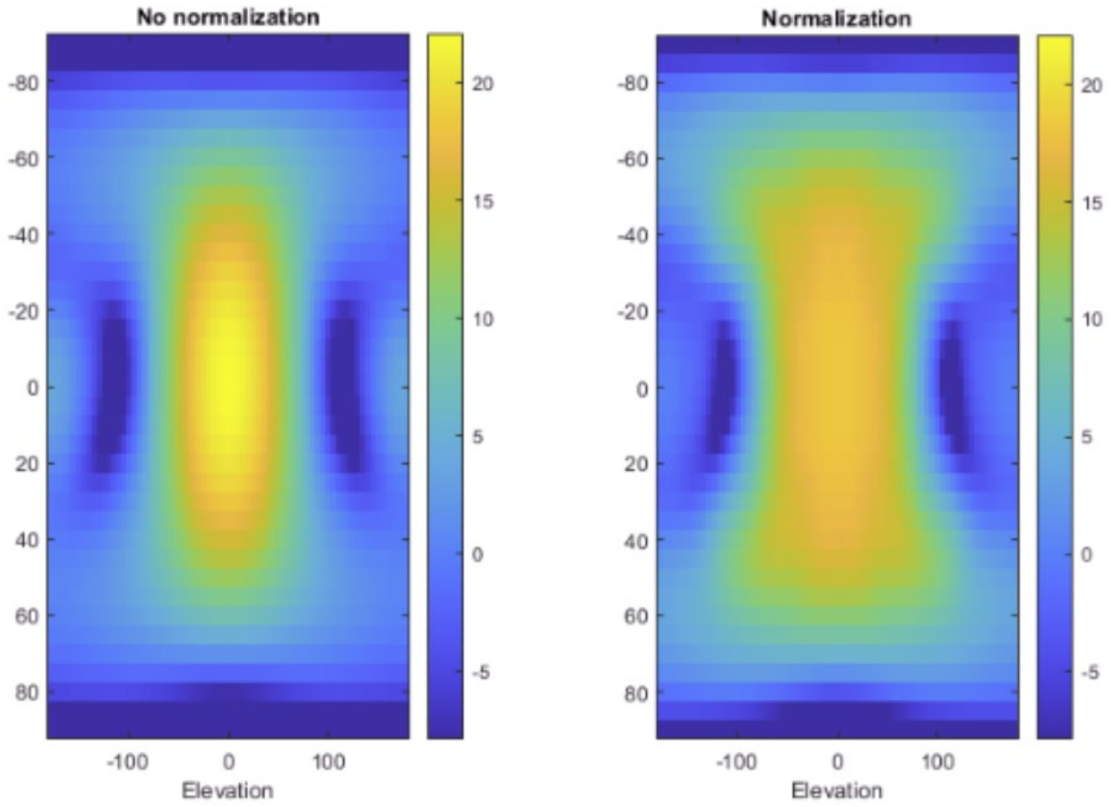
Target angle $(\theta, \phi) = (0,0)$

$(\theta, \phi) = (30,45)$



Max Gain

- Plotted:
  - Max gain in each angle
- With no normalization:
  - Max gain at boresight $= 12 + 10.1 = 22.1$ dBi
- With normalization:
  - Max gain at boresight $= 18.3$ dBi
  - Max gain at other angles more uniform



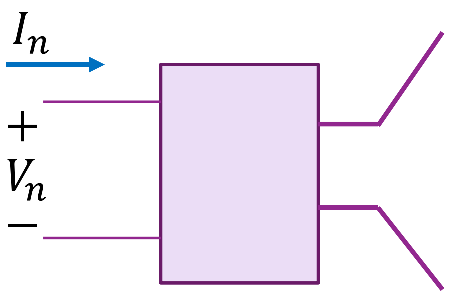
Proof Part 1: Analyzing in Current Domain

- Let $\boldsymbol{I} = [I_1, \dots, I_N]^T$ = vector of complex baseband current inputs to the antennas
- Consider electric field at angle $\Omega = (\phi, \theta)$ at far distance d
- Assume the electric field from a single current I_n is:

$$E(\Omega) = \frac{c}{d} A_E(\Omega) u_n(\Omega) I_n$$

  - c = some proportionality constant
- We know super-position applies for currents
  - This is a consequence of Maxwell's equations
- Hence with all N currents:

$$E(\Omega) = \frac{c}{d} A_E(\Omega) \sum_{n=1}^{N} u_n(\Omega) I_n = \frac{c}{d} A_E(\Omega) \boldsymbol{u}(\Omega)^T \boldsymbol{I}$$



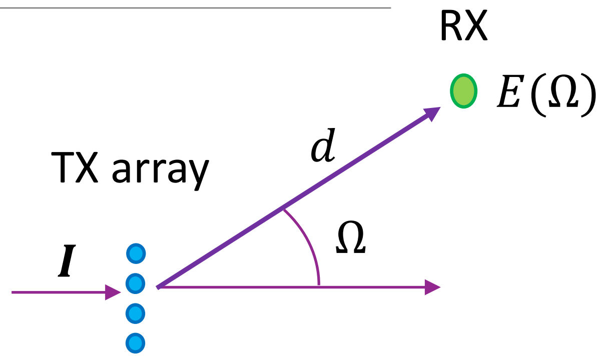
TX antenna n

Proof Part 2: Total Radiated Power

- From previous slide: Electric field is $E(\Omega) = \frac{c}{d} A_E(\Omega)\boldsymbol{u}^T(\Omega)\boldsymbol{I}$
- Hence, power intensity is $U(\Omega) = \frac{d^2}{2\eta}|E(\Omega)|^2 = C|A_E(\Omega)\boldsymbol{u}^T(\Omega)\boldsymbol{I}|^2$
  - $C = \frac{|c|^2}{2\eta}$, η =characteristic impedance
- Hence, the total radiated power is:

$$P_{tx} = \int U(\Omega)d\Omega = \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} U(\phi,\theta)\cos\theta \, d\theta d\phi = C\boldsymbol{I}^*\overline{\boldsymbol{Q}}\boldsymbol{I}$$

- Here $\boldsymbol{Q} = \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}|A_E(\Omega)|^2 \, \boldsymbol{u}(\Omega)\boldsymbol{u}^*(\Omega)\cos\theta \, d\theta d\phi$
- $\overline{\boldsymbol{Q}}$ =elementwise complex conjugate of $\boldsymbol{Q}$

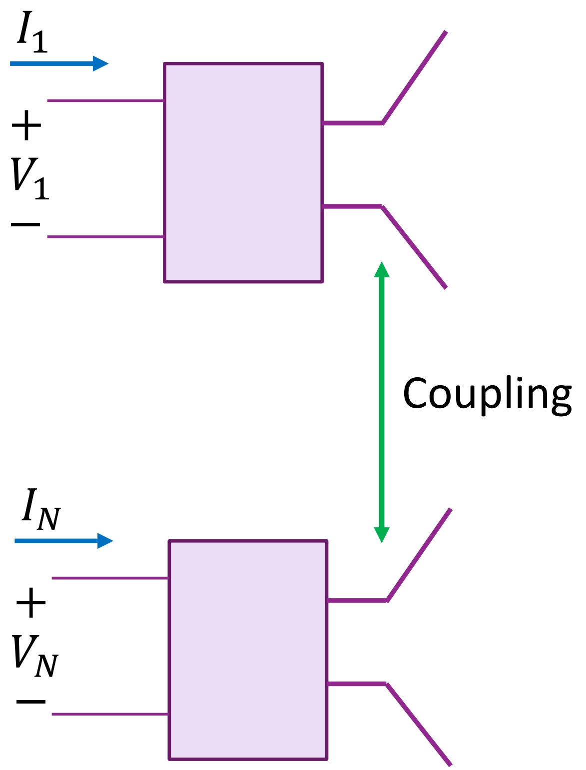


Proof Part 3: Array Resistance Matrix

- From previous slide we saw that :

$$P_{tx} = C\boldsymbol{I}^*\overline{\boldsymbol{Q}}\boldsymbol{I}$$

  - $\overline{\boldsymbol{Q}}$ can be computed from the integral of spatial signatures
- We know from network theory the power consumed is $\frac{1}{2}\boldsymbol{I}^*\boldsymbol{R}\boldsymbol{I}$
  - $\boldsymbol{R}$= resistance matrix of the array
- If the antennas are lossless, this power must be transmitted
- Hence $P_{tx} = \frac{1}{2}\boldsymbol{I}^*\boldsymbol{R}\boldsymbol{I}$
- Conclusions:
  - The matrix $\overline{\boldsymbol{Q}}$ is a scaled version of the antenna array resistance matrix
  - The matrix captures the coupling of currents and voltages between antennas



Proof Part 4: Computing the Channel

- Up to now we have shown:
  - Total transmitted power is $P_{tx} = C\boldsymbol{I}^*\overline{\boldsymbol{Q}}\boldsymbol{I}$
  - Radiation intensity at angle Ω is $U(\Omega) = C|A_E(\Omega)\boldsymbol{u}^T(\Omega)\boldsymbol{I}|^2$
- Define:
  - Power input vector: $\boldsymbol{s} = \sqrt{C}\overline{\boldsymbol{Q}}^{1/2}\mathbf{I}$
  - Normalized steering vector: $\boldsymbol{v}(\Omega) = A_E(\Omega)\boldsymbol{Q}^{-1/2}\boldsymbol{u}(\Omega)$
- With these definitions:
  - Total transmitted power is $P_{tx} = C\boldsymbol{I}^*\boldsymbol{Q}\boldsymbol{I} = ||\boldsymbol{s}||^2$
  - Radiation intensity at angle Ω is $U(\Omega) = C|A_E(\Omega)\boldsymbol{u}^T(\Omega)\boldsymbol{I}|^2 = |\boldsymbol{v}^T(\Omega)\boldsymbol{s}|^2$
- Hence $|\boldsymbol{v}^T(\Omega)s|^2$ is the power gain relative to free space propagation
- Therefore, channel can be modeled as $g_0\boldsymbol{v}^T(\Omega)\boldsymbol{s}$ is the free space channel